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13. WHAT CHARACTERISES HIGH ACHIEVING STUDENTS' MATHEMATICAL REASONING?

INTRODUCTION

This study investigates high achieving students' mathematical reasoning when given an unfamiliar trigonometric equation. The findings indicate that the students' way of thinking is strongly linked with imitative reasoning and only when they received some form of guidance, were they able to display flexible and creative mathematical reasoning.

Research Question

The purpose of this study is to investigate the reasoning that high achieving students' in upper secondary school display when they meet an unfamiliar trigonometric equation. The underlying motivation for the study, is the relationship between the socially constructed "high achievements" in school mathematics and the theoretical concept of "mathematical competence". For the purpose of this study, the author proposes that high achieving students are students who consistently get grades five and six in upper secondary school mathematics. But are high achieving students also mathematically competent students? Are the two terms synonymous? Research by Lithner (see for instance 2000, 2003 and 2008) indicate that even high achieving students make use of superficial reasoning when given an unfamiliar mathematical task. Niss & Jensen (2002) dissect mathematical competency into eight, distinct and clearly recognizable competencies: thinking mathematically, posing and solving mathematical problems, modelling mathematically, reasoning mathematically, representing mathematical entities, handling mathematical symbols and formalisms, communicating in, with and about mathematics and making use of aids and tools. Other frameworks, e.g. NCTM (2000) define mathematical competence similarly. However, mathematical cognitive activity is incredibly complex. Every investigation of mathematical understanding will have to be in some ways simplified (Niss, 1999). So although this study is motivated by the possible discrepancy between high achievements and the term mathematical competency in upper secondary school mathematics, the aim here is to capture some key aspects of high achieving students' reasoning structure when working with a mathematical problem. Not to give a complete description of high achieving students' mathematical understanding and mathematical thinking vis-à-vis certain mathematical concepts.

B. Sriraman and K.H. Lee (eds.), The Elements of Creativity and Giftedness in Mathematics, 193–215.

In this study, the author hopes to qualitatively characterise the mathematical reasoning of three high achieving students when they meet a mathematical problem and the following two research questions will be investigated:

- Is it true that high achieving students display superficial reasoning when given an unfamiliar trigonometric equation?
- What characterises the students' mathematical reasoning when given an unfamiliar trigonometric equation?

Hiebert (2003) argues that students learn what they are given an opportunity to learn. If there is a lack of focus on mathematical reasoning, mathematical thinking and problem solving in the students' learning environment, it is the author's contention that it may not be unrealistic to expect even high achieving students in some situations to focus on surface and not structural features of mathematical problems. Trends in International Mathematics and Science Study, TIMSS, Advanced 2008 (Mullis et al., 2009) and the PISA + study (Kirsti Klette et al., 2008) show that there is a lack of focus on problem solving and mathematical reasoning in both upper and lower secondary school in Norway. Students are rarely asked to explain their answers and communicate mathematical arguments to others. Instead, the primary activities in the classroom are direct instruction from the teacher and students working on problems on their own. Furthermore, the problems the students are working with are, according to Klette et al. (ibid), not stimulating problem solving skills. If students learn what they are given an opportunity to learn, do even high achieving students resort to superficial reasoning when given an unfamiliar mathematical task? Or are they able to, in many ways in spite of their learning milieu, identify and focus on the structural features of the problems and display mathematically correct reasoning? The first research questions sets out to determine whether or not there are high achieving students who actually do display superficial mathematical reasoning in upper secondary school. Once this has been answered, a more thorough investigation of the mathematical reasoning displayed is needed. The second research question looks more closely at the quality of the mathematical reasoning displayed by the students when they meet an unfamiliar task, to see if there are certain characteristics that are associated with the students' mathematical reasoning.

Literature Review

Algorithms are a key component of mathematics. They can not only serve as the basis for mathematical understanding, but they can also relieve the cognitive demands of complicated calculations. Even professional mathematicians use algorithms and fixed procedures when dealing with routine calculations. However, algorithms and procedures are just one small part of mathematics. Halmos (1980) states that problem solving is the heart of mathematics which in many ways is supported by Freudenthal (1991) who claims that mathematics is an activity of discovering and organizing of content and form. In a learning environment, algorithms and procedures need to be supplemented with other aspects of mathematics such as problem solving activities and deductive proofs. A narrow focus on algorithms and routine tasks can limit the students' ability to use mathematics. McNeal (1995) and Kamii & Dominick (1997) argues that students when working with algorithms, tend to focus on remembering each step in the algorithm and not the underlying mathematical structures. Pesek & Kirshner (2000) states that instrumental instruction can interfere with later relational learning. Furthermore, not only can a narrow focus on algorithms and skill to some degree hinder learning of mathematics, but the mathematics curricula in most countries emphasize problem solving on its own as an important aspect of mathematics. The NCTM Standards states that reasoning and problem solving are key components of mathematics and we find similar statements in the Norwegian mathematics syllabus (KD, 2006).

Traditional mathematics teaching emphasizes procedures, computation and algorithms. There is little attention to developing conceptual ideas, mathematical reasoning and problem solving activities. The result is that students' mathematical knowledge is without much depth and conceptual understanding (Hiebert, 2003). These findings are also seen in Selden et al.'s (1994) study where students with grades A and B struggle with non routine problems. Selden et al. concluded that the students possessed a sufficient knowledge base of calculus skill and that the students' problem solving difficulties was often not caused by a lack of basic resources. Instead, they say, traditional teaching does not prepare students for the use of calculus creatively. Lithner (2003) and Schoenfeld (1985) show how many of the students, even high achieving students, try to solve problems using superficial reasoning. A possible hypothesis which could explain this phenomenon is seen in Cox (1994), where the author argues that first year students in universities are able to get good grades by focusing on certain topics at a superficial level, rather than develop a deep understanding. It is important at this stage to clarify that it is not the author's claim or intention to argue that high achieving students are not capable of becoming mathematically competent students or that high achieving students in general are not mathematical competent students. However, there are certain indications that students can get good grades in school, in spite of certain shortcomings vis-à-vis the concept of mathematical competence as defined by Niss and Jensen (2002) and NCTM standards.

Much research within mathematics education has focused on learning difficulties regarding mathematical understanding and in a broader sense, my research question are part of a greater, more fundamental issue in mathematics education. Skemp (1976) defined this issue as the dichotomy between relational understanding and instrumental understanding. Instrumental understanding consists of a number of fixed and specific plans or strategies for solving specific tasks. The students lack an overall understanding of the relationship between the individual stages and the final goal of the exercise. To learn a new way to solve a particular branch of problems, as a *"way to get there",* the learner is dependent on external guidance. Relational understanding, on the other hand, is defined as: *"[it] consists of building up a conceptual structure from which its possessor can produce and unlimited number of plans for getting from any starting point within his schema to any finishing point."* (ibid). The knowledge and understanding becomes the goal in itself, not necessarily successfully solving a particular problem. The plans are no longer fixed and immediately tied to a particular class of problems. Others, such as Ausubel (1962) and Hiebert & Lefevre (1986) outlines similar dichotomies: meaningful vs. rote learning and conceptual vs. procedural knowledge respectively. Relational understanding, meaningful learning and conceptual knowledge are all characterised by the fact that new knowledge is related and connected to other existing schemas.

A related issue in dealing with difficulties in mathematical understanding is students' tendency to view mathematics almost exclusively as a collection of processes or procedures. Evidence suggests that the flexibility to view objects as both a process and a concept is vital to future success (Gray & Tall, 1991). This cognitive conflict is similarly described via versatile and adaptable mathematical knowledge within the context of algebra (Sfard & Linchevski, 1994). Versatile knowledge is being able to view mathematical expressions in many different ways. Adaptable knowledge is being able to view a mathematical expression in an appropriate way. Versatility refers to the different ways of solving a problem and how each of those strategies are carried out. Adaptability refers to choosing the most appropriate strategy for the problem at hand. Sfard & Linchevski states that both versatility and adaptability is necessary to fully succeed in algebra. The extensive procedural focus on mathematics is characterised as a reduction of complexity of mathematical concepts, processes and mathematical thinking. Research shows that students, teachers, textbook writers etc, in order to satisfy tough curriculum goals focus on algorithmic thinking and not deep mathematical reasoning (Lithner, 2005).

CONCEPTUAL FRAMEWORK

In the field of mathematics education, the term mathematical reasoning is often used without an explicit definition, under the assumption that there is a universal agreement on its meaning (Yackel & Hanna, 2003). Lithner (2006) claims that reasoning is often implicitly seen as a process characterised by a high deductive-logical quality, frequently in connection with formal mathematical proofs. However, in this study the students are in upper secondary school and such a strict definition of reasoning is not appropriate. Thus, a more inclusive definition of the term reasoning is used in this study. In Merriam-Webster's online dictionary, reasoning is defined, among other, as "the drawing of inferences through the use of reason". Reason is then defined as "a statement offered in explanation or justification". So mathematical reasoning is, for the purpose of this study, the line of thought adopted to produce assertions and reach conclusions in task solving (ibid). Or, the arguments produced to convince one self and/or others of the truth of an assertion. A line of thought might be mathematically incorrect or flawed, as long as it makes some kind of sense to the reasoner itself.

Harel (2008) defines mathematical activity as a triad of concepts: mental act, way of understanding and way of thinking:

> "A person's statements and actions may signify cognitive products of a mental act carried out by the person. Such a product is the person's way of understanding associated with that mental act. Repeated observations of one's way

of understanding may reveal that they share a common cognitive characteristic. Such a characteristic is referred to as a way of thinking associated with that mental act".

Mental acts are basic elements of human cognition, such as interpreting, inferring, proving, generalizing etc. Although mathematical reasoning involves numerous mental acts, in this study mathematical reasoning itself is a mental act. It is the cognitive process of convincing oneself and/or others of the truth of an assertion.

In this study, the author intends to investigate what characterises high achieving students' mathematical reasoning. By looking at the students' arguments, solutions and written work, a pattern characterising their work could appear. Characteristics of their way of understanding might give some insight into their way of thinking. There is an important difference between behaviour and cognition. This dichotomy between way of thinking and way of understanding, is also seen in Lithner's (2006) view of reasoning as both a thinking process and the product of that process. The product of the thinking processes, the way of understanding, we can observe as behaviour, but whatever inferences we make regarding the underlying cognitive processes, will still be, to some degree, speculative. In this study, mathematical reasoning is a mental act and the purpose is to investigate the students' way of thinking. This is done by looking more closely at the students' way of understanding.

To further investigate the mental act of mathematical reasoning and the characteristics of the students' way of understanding, the author have chosen a framework by Lithner (2006) that allows me to qualitatively classify and assess specific aspects of the students' mathematical reasoning. The framework is built up using specific mathematical examples and well defined concepts describing mathematical reasoning. It is based on empirical data and it describes mathematical reasoning in general and is not tied up to a particular set of problems or themes. Here, two main classes of reasoning are defined: creative reasoning and imitative reasoning:

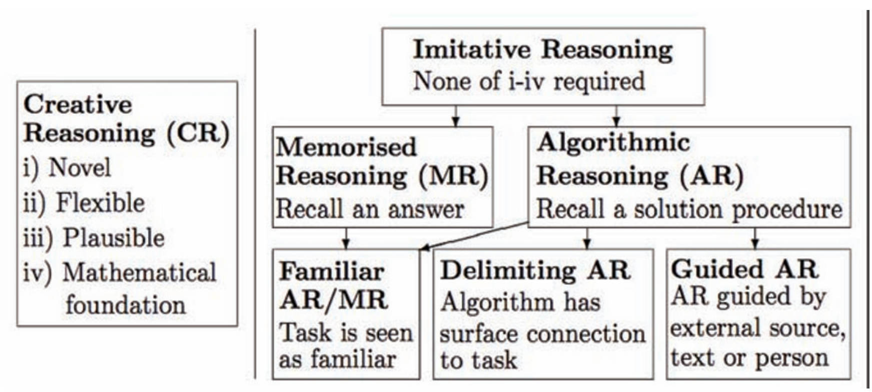

Figure 1. Creative and imitative reasoning (Lithner, 2008).

CREATIVE REASONING

The basic idea of creative reasoning, or creative mathematically founded reasoning as it is also referred to in the framework, is the creation of new and reasonably well-founded task solutions. Not necessarily geniality or superior thinking. For the reasoning to be called creative reasoning, two conditions must be met (Bergqvist, 2007):

- The reasoning sequence must be new to the reasoner (novelty)
- The reasoning sequence must contain strategy choices and/or implementations supported by arguments that motivates why the conclusions are true or plausible (plausibility), and are anchored in intrinsic mathematical properties of the components involved in the reasoning (mathematical foundation).

The definition of creative reasoning is very similar to creativity in general. Sternberg & Lubart (1999) define creativity as the ability to produce original and useful work. Sriraman (2009) raises the objection that many mathematicians would object to the criteria of usefulness, as a lot of work in mathematics do not have immediate implications for the “real world”. However, in this study, the term usefulness is seen as a correct solution of a mathematics task. The mathematical reasoning is plausible and based on intrinsic mathematical properties. Not whether or not it has implications or uses in the “real world”. Creative reasoning is therefore a subset of the general term creativity and the terms novel and plausible are analogous to original and useful. Furthermore, the originality and novelty of the mathematical reasoning, is relative to the reasoner. What might be trivial routine for a mathematics professor, could be an original and novel solution to a problem for an upper secondary mathematics student. The reasoning sequence must be new to the reasoner, not necessarily new to the rest of the mathematics community.

Creative reasoning does not imply strict logical deductive reasoning. Even though it is normal to distinguish a proof from a guess vis-a-vis mathematical reasoning (Polya, 1954). The value of the reasoning of a proof is based on its correctness or logical rigour. Students however, unlike mathematicians, engineers, economists etc, can afford to guess, take chances and not always give the correct answer to every problem or exercise. Creative reasoning therefore distinguishes a guess from a more reasonable guess, and not a guess from a proof. To determine whether or not a sequence of reasoning is creative mathematically founded, the following criteria must be fulfilled: novelty, flexibility, plausibility and a sound mathematical foundation.

Novelty refers to the fact that a new, to the reasoner, sequence of reasoning is created or a forgotten sequence is re-created. If an answer or solution is imitated, it is not considered to be creative mathematically founded reasoning. Second, the reasoning must be flexible. This implies the ability to utilize different approaches and adaptations to the specific problem. The student is not fixed on one specific strategy choice or sequence of reasoning that hinders progress. Plausibility means that there are arguments supporting the strategy choice and explains why the conclusions are true or plausible. Last, the arguments are based on intrinsic mathematical properties; the arguments are based on a sound mathematical foundation.

The arguments used to show that an answer to a problem or exercise is correct can be based on sound mathematical properties or less sound mathematical properties. However, before sound and less sound mathematical properties can be defined we need to establish the mathematical components which we deal with when solving problems. The framework defines the following components relevant for solving mathematics exercises and problems: Objects are the things that one is doing something with. This could be numbers, functions, variables etc. Transformations is what is being done to the objects. For instance adding to real numbers. Last, is the concept which is a central mathematical idea built on a set of objects and transformations (e.g. the function concept). These components have certain mathematical properties and the framework separates intrinsic and superficial properties.

An intrinsic mathematical property of an exercise is a property that is relevant for how you solve the exercise. This means that an intrinsic mathematical property is central to a particular context and in a particular problem. A surface property, on the other hand, has little or no relevance for how a given exercise can be solved. In each task, there are potentially numerous both intrinsic and surface properties. The relevancy of a mathematical property depends on the context. For instance, in deciding if or is larger, the size of the numbers is a surface property, while the quotient captures the intrinsic property (Lithner, 2008). Another example is naive empiricism (Schoenfeld, 1985) in an attempt to bisect angels. The visual appearance of angles is a surface property, while the formal congruency of the triangles in the construction is the intrinsic property.

IMITATIVE REASONING

Imitative reasoning is a term that describes several different types of reasoning which are based on previous experiences, but without any attempts at originality. This means that students try to solve problems and exercises by copying textbook examples, earlier task solutions or through remembering certain algorithms. Imitative reasoning is in many cases a superficial sequence of reasoning, not grounded on intrinsic mathematical properties, but rather on surface properties. The students chose their strategy for solving the problems on superficial properties they recognize from earlier experiences and not on intrinsic mathematical properties. From empirical research, imitative reasoning has been classified further into subcategories, where the two main categories are memorized reasoning and algorithmic reasoning.

Memorized Reasoning

Memorized reasoning is determined by two conditions: first, the strategy choice is founded on recalling a complete answer by memory. Second, implementing said strategy choice consists only of writing it down. For instance, remembering each step of a proof and writing it down is memorized reasoning. Of course, a common mistake when employing memorized reasoning is that the different parts of a solution can be written down in the wrong order since the parts do not depend on each other and the reasoning is not based on intrinsic properties.

Algorithmic Reasoning

An algorithm is a set of instructions or procedures that will solve a particular type of problem. For instance, the chain rule for finding the derivative of a composite function.

Algorithmic reasoning is determined by two conditions: first, the strategy choice is founded on recalling an algorithm that will guarantee that a correct solution can be reached. Second, implementing the strategy consists of trivial transformations. The formula for solving a quadratic equation is an example that illustrates the difference between algorithmic reasoning and memorized reasoning. In the latter case the exact same equation and corresponding solution would be written down from memory. In the former case the algorithm, or formula, would be applied to this specific equation. The student would not recall the entire solution to the equation, but rather remember the algorithm and know that it would give a solution.

Algorithmic reasoning is a reliable method for solving problems when the student knows exactly what to do and why the chosen algorithm is appropriate. Even professional mathematicians use algorithms when solving routine problems. The use of algorithmic reasoning in itself is not an indication of a lack of understanding as it saves time and reduces the risk of miscalculations. The key here is the mathematical properties on which the algorithmic reasoning is based on. In many cases students use algorithmic reasoning in problematic situations, which indicates that it is based on superficial and not intrinsic mathematical properties.

METHODS

Procedures

The empirical data was collected from three clinical task based interviews. In each interview, the students were given a specific trigonometric task designed by the author and asked to solve it while they were "thinking aloud". Each interview lasted for approximately 30 minutes. Before each interview, a short, informal conversation between the student and the author took place. The objective of the short, informal conversation was to create a more comfortable environment and situation for the student. At the start of the interview, the following monologue, recommended by Ericsson & Simon (1993) was given by the researcher in order to initiate the student's think aloud talk:

> "Tell me EVERYTHING you are thinking from the time you first see the question until you give an answer. I would like you to talk aloud CONSTANTLY from the time I present each problem until you have given your final answer to the question. I don't want you to try to plan out what to say or try to explain to me what are you saying. Just act as if you are alone in the room speaking to yourself. It is most important that you keep talking. If you are silent for any long period of time I will ask you to talk (p. 378)."

The interview was separated into two parts. The first 10–15 minutes, the author stayed silent and only reminded the subjects to keep talking if they stayed silent for extended periods of time. If the students struggled with the task given and showed signs of giving up, the author gave them a similar, but simpler task and asked them if they could solve the new task. After working for a few minutes on the new task, the author then asked them if they now could go back and solve the original task. The final 15 minutes of the interview, was less structured and the author asked more direct questions. Trying to get the students to justify and explain what they were doing and why they were doing it.

Participants

Three students in grade 13 who are all taking an advanced mathematics course in a local upper secondary school were selected for this study. However, it was not the author who selected the students to be interviewed. Instead, the mathematics teacher who was teaching the advanced mathematics course was asked to select 2–4 students which she deemed to be high achieving. The author wanted the teacher to select the students for the study, as this would, in a more general sense, give access to students who the Norwegian educational system classifies as high achieving students. The author had no other criteria set forth to the teacher, other than that the students had to be considered consistent high achievers in mathematics. The author did not ask the teacher to select what she would call typical or atypical high achieving students. There were two reasons for this. First, the author wanted to see which students the teacher, when given few restrictions, would classify as high achieving students. Second, it might have been difficult to find high achieving students if there were several restrictions.

Tasks

The task given to the students, was the trigonometric equation in which the students were asked to find a:

$$\sin x + \cos x = a$$

The task was chosen for several reasons. First, the students in grade 13 are quite familiar with trigonometric equations. In the textbook, there is a large section devoted entirely to trigonometric functions and a subsection which focus specifically on the equation. It would therefore be reasonable to expect high achieving students to have the necessary domain knowledge to solve the trigonometric equation. Second, the task is designed in such a way that it can be solved in a multitude of ways. Third, compared to the tasks given in the textbook and tasks given by the teacher to the class, the task is unusual. Not only because the answer is an interval and not a single value, but also because it contains both the variable x and the parameter a. Furthermore, in the textbooks, c is given as an integer. In the task in the study, c is an unknown parameter. This presumably creates a problematic situation for the students. They have the necessary domain knowledge to solve the problem, but may not know of any immediately available procedures or algorithms that will solve it. This opens up for both flexible and creative reasoning when trying to solve the equation. In this article, the terms problem, task and equation will be used interchangeably about the trigonometric equation the students' tried to solve during the interview.

If the students looked like they were struggling with the first task, a prompt was given. This is also a trigonometric equation, but less complex. Here as well, the students were asked to find *a*:

$$\sin x = a$$

This task was designed with two considerations in mind. First, the task is significantly easier than the original task. There are fewer components in the task and, as such, it might be easier to notice the structural aspects of the task than in the first task given. So the prompt was designed in order to help the students by reducing the complexity of the tasks. Second, it also allowed the author to see if the students were able to generalize their reasoning from the simpler task to the more complex task.

ANALYSIS

The data material consisted of the transcribed interviews and the written work the students produced during the interview. The interviews were transcribed by the author. As the research questions in this study deals with characterizing students' mathematical reasoning, the transcriptions were primarily focused on verbal and written mathematical communication. Such as arguments, guesses, assertions, conjectures etc directly related to mathematics which were produced in written or oral form during the interview. Although other aspects such as body language, type of interaction between the author and the student, tone of voice etc obviously play an important part in the students' behaviour and may say something about the students' mathematical reasoning in general, in this study the focus is the students' mathematical argumentation and justification. The explicit mathematical reasoning they display when they meet an unfamiliar mathematical problem.

The analysis consisted of two parts; first, identification of each reasoning sequence and then classification of each reasoning sequence according to the framework:

1. Lithner (2008) proposes that a reasoning structure is carried out in four parts. A task is met, a strategy choice is made, the strategy is implemented and a conclusion is obtained. To identify separate reasoning sequences, the strategy choice and conclusion of each reasoning sequence was identified in the transcripts. Strategy choice is seen in a wide sense here. Strategy ranges from local procedures to general approaches and choice includes recall, choose, construct, discover, guess etc. A conclusion is reached after the strategy has been implemented. The conclusion is simply the product of the implementation of a certain strategy and it can be both incorrect and/or incomplete. For each student, there might be one or several attempts at solving the task.
2. After the individual sequences was identified, each reasoning sequence was then classified according to the framework presented earlier. This was accomplished by first classifying the reasoning sequence as either creative reasoning or imitative reasoning. If the reasoning sequence was classified as imitative reasoning, further analysis is carried out to determine whether the reasoning sequence is algorithmic or memorized.

METHODOLOGICAL ISSUES

Validity

Regardless of which epistemological position one takes it is widely accepted that there is a need for some form of measurement of validity in qualitative research (Ritchie & Lewis, 2003). For the purpose of this particular study, a pragmatical and practical view of validity will be adopted. Research, both quantitative and qualitative, is a human experience prone to the same mistakes as every other human activity.

The primary concern in this study, is internal validity, which relates to the extent the research correctly map or document the phenomenon in question (Hammersley, 1990). Is the author really investigating what the author claims to be investigating? In this study, this question is related to whether or not the author is able during the interview, and in the following analysis, to characterize the students' mathematical reasoning. Every interview will have epistemological conflict between the need for complete or rich data and the need for minimizing interference (Clement, 2000). As a compromise, the interview session in this study consisted of two parts. During the first part, the students were asked to think aloud while they tried to solve a trigonometric equation. The students talked freely with little or no intervention from the author. In the second part, the author engaged more actively in the interview and probed further, in order to get the students to explain in greater detail their mathematical reasoning. The structure of the interview was designed in order to provide valid and rich data, respectively.

Reliability

Reliability describes the replicability and the consistency of results. Hammersley (1992) refers to reliability as: "...the degree of consistency with which instances are assigned to the same category by different observers or by the same observer on different occasions." Although qualitative researchers, whether they are constructivists or positivists, do not calculate interrater reliability (Ballan, 2001), other steps can be taken to improve reliability. In this article, both the collection and analysis of the data was carried out by the author. To improve reliability, extensive excerpts from the interviews, which the analysis is based on, are given. The methods used to gather and analyze the data are also explained in detail.

RESULTS

Here, several examples of individual reasoning sequences will be given and analyzed at two points during the interview session: when they first begin working on the problem and when they are given the prompt and solve the problem. For each episode, a description of what the student was doing will be given. Then the reasoning structure will be outlined and commented on. Each of the reasoning sequences can be said to be the students' way of understanding and the recurring characteristics of their way of understanding, says something about the students' way of thinking.

Attacking the Problem

An interesting aspect of students' mathematical reasoning, is how they first attack a problem. How do the students approach a new problem when they are not given any specific instructions regarding how the problem is to be solved. Here, a description and interpretation of what the students first did when they saw the problem are given.

– **Alf**

Description

[The author gives Alf the task.]

Alf: "Ok, so here we have cosine and sine and we need to get this into a regular cosine function. Let's see. [Looks in his textbook for about 15 seconds]. I don't remember which chapter this was. Yes. Now I remember. It was about harmonic equations. Let's see. Ok. I see this is a sine and cosine function that can be transformed into a tangent function. Sin x divided by cos x. [Does some calculations on his work sheet]. Same as tan x. Plus one equals *a*. So that is tan x plus one equals *a*. That is what I was supposed to find?"

Author: "You can find a numerical value for *a* in the task".

Alf: "I can? But then I need to find tangent *a x*, I need to invert it. I can't find *x*. How can I find a numerical value if I have two variables? I have two unknowns. [Alf is quiet for 10 seconds]. Let's see. If I am going to find *a*, I need to find *x* first. But that makes no sense."

Interpretation

Alf's first reasoning sequence when he begins working on the problem is quite clear:

1. Strategy choice: divide each term by cos x and use that $\tan x = \frac{\sin x}{\cos x}$ to simplify the equation. Then, invert both sides of the equation and find *x*.
2. Strategy implementation: Alf divides each term on the left side of the equation by cos x. However, he forgets to divide the term on the right side by cos x.
3. Conclusion: The equation 1 + tanx = a can not be solved.

The procedure Alf first chose, transforming an equation with a sine and a cosine term into an equation with a tangent term, is, as he said, mentioned several times in the students' textbook. Both in the form of worked examples and as similar tasks. Alf recognizes the general structure of the equation and assumes it can be solved as the tasks and examples he has seen in the textbook. The argument he presents, is that he has seen this type of equation several times in the textbook. It is a familiar algorithm. So, it is reasonable to conclude that Alf's first attempt at solving the task is imitative in nature and, more specifically, algorithmic reasoning.

The mistake Alf made implementing the strategy, was apparently a minor and insignificant, as he corrected it later during the interview without the author pointing it out. It is also not unreasonable to expect that Alf would have drawn the same conclusion even if he had implemented the strategy correctly. This is seen when he states that he can't find a number if he has two variables. The two variables or unknowns are *x* and *a*. Even if he had implemented the strategy correctly, he would have ended up with what Alf calls "two variables".

– **Anna**

Description

[The author gives Anna the task.]

Anna: [Reads the task instructions and is quiet for a few seconds]. "Ok, I need to find some connection between sine and cosine, but i don't remember the formulas by heart. [Anna looks in her textbook for about 15 seconds]. Ok, if I divide each term by cos x, then I might end up with tan x here on the first part. [She proceeds to divide each term in the equation with cos x, writes down the answer on paper.] So I have one and a divided by cos x. That doesn't help me much."

Author: "Why not?"

Anna: "I think I have two unknowns here. Both tangent and cosine. [Anna is quiet for the next 15 seconds.]

Interpretation

Anna's first reasoning sequence is similar to Alf's first reasoning sequence:

1. Strategy choice: divide each term by cos x and use that $\tan x = \frac{\sin x}{\cos x}$ to simplify the equation. Then, invert both sides of the equation and find x.
2. Strategy implementation: divides each term on both sides of the equation by cos x.
3. Conclusion: the equation $1 + \tan x = \frac{a}{\cos x}$ can not be solved.

Anna's first attempt at solving the task is, as Alf's first attempt, algorithmic reasoning. She doesn't remember the procedure in full, but after looking in her textbook for a few seconds she remembers it and applies it to the task. She quickly concludes, however, that the procedure didn't solve the task and she has to start over. The only significant difference in Alf's and Anna's reasoning structure, is that Anna implements the strategy correctly. However, the conclusion is, as Alf's conclusion, that the resulting equation can not be solved, as she says she has two unknowns there. Presumably, both Anna and Alf wanted to invert both sides of the equation and find a numerical value for x. This is how the procedure is described in the textbook.

– **Hege**

Description

[The author gives Hege the task.]

Hege: "I don't remember this. It's been a while since we were working on this. Sine x plus cosine x equals [uniteligeble], we two unknowns here. I need to somehow combine sine x and cosine x into one expression. I don't remember anything of this." [She start looking through her textbook. She is quiet for about 30 seconds.]

Author: "What are you looking for?"

Hege: "I don't know. I need to find some formula to combine cosine x and sine x. [She looks in her textbook for the next minute, then puts it down.] It wasn't there. I'm trying to find some formulas for the sum. But I can't do it when I don't know what x is."

Author: "What are you supposed to find here?"

Hege: "An unknown, but I don't know how to do it."

Interpretation

Hege's reasoning structure is less clear than the previous two. She looks for a trigonometric identity or procedure in the textbook that can transform the given task into an equation with just one trigonometric expression, but says she didn't find what she was looking for. Although she quickly gives up, the reasoning structure can be formulated as following:

1. Strategy choice: simplify the equation by finding a formula or procedure that can combine two trigonometric expressions into one trigonometric equation.
2. Strategy implementation: look for a formula or procedure in the textbook.
3. Conclusion: an appropriate formula or procedure was not found and the task can not be solved.

Based on the description of Hege's first attempt at solving the equation, it is difficult to know exactly what procedure or formula she is looking for. If the author were to speculate, it seems she vaguely remembers a procedure or formula that can transform this equation from having two trigonometric terms into an equation with just one trigonometric term. Even though Hege's attempt at solving the equation is neither as sophisticated nor as fruitful as the other two students' attempts, it is similar in nature. As Anna and Alf, Hege's first attempt is clearly algorithmic reasoning as she looks for a specific procedure or formula that can help her solve the equation. The main difference between Hege and the other two students, is that Hege does not remember or know the exact procedure.

The Prompt and Solution

Another interest of the author, which would shed some light on the students' mathematical reasoning, was their behaviour when given the prompt. The purpose of the prompt was two fold. First, it served as an implicit hint. It gave some insight into how the original equation could be solved. Second, the author wanted to see how the students generalized the properties of the prompt to the original equation.

– **Alf**

Description

[The author gives Alf the prompt.]

Alf: "Ok. This is the sine graph. I can just draw it." [Alf draws the graph of the sine function on a piece of paper and is quiet for about 15 seconds]

Author: "Can you say something about *a*?"

Alf: "*a* is an interval. It's between -1 and 1." [Alf writes down $a = \langle -1,1 \rangle$ next to the sine graph.]

Author: "so if you now go back to the original problem..."

Alf: "ok, but the tangent function is completely different. It goes like this." [Alf draws the graph of the tangent function next to the graph of the sine function.]

Author: "What if you look at the original equation."

Alf: [Alf is quiet for 10 seconds]. "I can transform it into a sine function and then find what *a* can be. As an intervall. It is dependent, but it must be within some range. It can. Wait a minute. *A* must be a function of [unintelligible] oscillation with sine. I can see that now. [Alf does some calculations on the piece of paper]. Then we have *a* squared plus *b* squared. This is it. *C* is the square root of two. Then we have to find the others as well. That means. But *c* is also one. So that is easy. It is the square root of two multiplied by sine x. Plus *b* over *a*. Which is one. Plus *d*, which we don't have to find, because that is just the equilibrium position which is zero. So *d* equals zero. [Alf continues to write down his solution on the piece of paper]. Ok, since the peak amplitude is the square root of two, that means *a* is between the square root of two and minus the square root of two."

Author: "Why is that?"

Alf: Because I made it into a sine function. Which means I have a graph that represents *a* here."

Interpretation

When given the prompt, Alf immediately says that this is the sine function and that a must be between -1 and 1. After a bit of guidance, he identifies the original equation of being similar in nature to the prompt. He knows that *a* is an interval in both cases. The generalization from the simpler to the more complex case is mathematically correct and he identified the structural similarities of the two equations. He furthermore shows a flexible understanding of the equation when he says that it is a function. The problem given is no longer just an equation with two variables, as Alf stated earlier, but *a* is a function of *x* as well now. This allows him to calculate the range of the function. The reasoning structure as he solves the equation is as following:

1. Strategy choice: The equation is also a function. The left side of the equation can be transformed into a sine function using $a \sin kx + b \cos kx = A \sin(kx + c)$. Find *a* by finding the range of the function.
2. Strategy implementation: Alf calculates $A = \sqrt{a^2 + b^2} = \sqrt{2}$, $k = 1$ and, wrongly, that $c = \frac{a}{b} = 1$. He concludes that the equilibrium position is zero. Giving him that $\sin x + \cos x = \sqrt{2} \sin(x + 1)$. The peak amplitude is $\sqrt{2}$, so $a = \left[-\sqrt{2}, \sqrt{2} \right]$.
3. Conclusion: *A* is an interval between minus the square root of two and the square root of two.

Even though Alf made a minor mistake in calculating c, this reasoning structure is to some degree creative in nature. The calculations Alf carried out were algorithmic and procedural, but looking at the equation as a function and evaluating its range is flexible, plausible and based on mathematical properties. Three of the four criteria needed for the reasoning to be classified as creative. The last of the criteria, novelty, is more difficult to evaluate. The author can not claim with certainty that the reasoning sequence was new to Alf, but the fact that he needed to see the prompt to make the necessary connections and initiate this reasoning sequence, indicate that it could be novel. Either as entirely new to the reasoner or as rediscovering a forgotten reasoning sequence.

An important point that needs to made, is that it was only after seeing the prompt Alf discovered the structural properties of the original equation necessary to apply his strategy choice. It was only after seeing the simpler equation he noticed that *a* had to be an interval and that he could look at the original equation as a function. From there, he could apply a familiar procedure in order to find the function's range. When Alf first began working on the equation, he didn't seem to investigate more closely the nature of *a* or what he was asked to find. Instead, he just looked at the equation, applied a familiar procedure and concluded it could not be solved as the equation contained two variables. The author finds it reasonable to conclude that Alf first focused on the surface properties of the equation and only after seeing the prompt, did he focus more closely on the structural features of the equation.

– **Anna**

Description

[The author gives Anna the prompt.]

Anna: [Quiet for 20 seconds]. "I have two unknowns. If it had been a number instead, for example two [pointing to *a*], then I could just say that *x* is sine inverted of two"

Author: "Ok".

Anna: "But I wouldn't get anything. Not now anyway."
Author: "Why not?"
Anna: "Because sine can not be greater than one. [Quiet for 5 seconds]. I am very uncertain about this. When I have two unknowns."
Author: "What is *a*? Is it an integer?"
Anna: "It can be many. *A* is a variable, right?" [Quiet for 15 seconds].
Author: "Remember to keep talking."
Anna: "I don't know. It's all very difficult now."
Author: "You said *a* could be many. What can *a* be?"
Anna: "It can be between one and minus one. I don't think I can find a more accurate answer than that."
Author: "What if you look at the original equation?"
Anna: "[Quiet for 10 seconds]. Yes, I don't know. It's been a while since we worked on this. Secondly, when there are so many unknowns. I don't know. What am I supposed to find here? Is it a number, is it an interval is it an expression."
Author: "What did you find in the other task?"
Anna: "That *a* had to be between one and minus one."
Author: "What is that?"
Anna: "It is an interval. [quiet for a few seconds]. Ok, it is exactly the same task. There is an interval here as well. Only difference is that here we have sine x plus cosine x and here it is just sine x. So if I can use this way of solving this, then I get. But then I get one and minus one."
Author: "It's the same interval in both equations?"
Anna: "Yes, it must be. If sine x plus cosine x, then both of them can be one. So one plus one equals two. But they can't be one simultaneously. If we look at the unit circle, [draws the unit cirlce] if cosine x is one, then sine x is zero. So maybe what I found is correct after all."
Author: "You could see if it is correct."
Anna: "Ok, if I try 40 degrees. [makes the necessary calculations on a calculator]. Ok, that is not right. I got 1.41. [Quiet for 25 seconds]. I must have misunderstood. [She picks up the calculator]. If I put it in here. If i put in y eqauls sine x plus cosine x. And set x from zero to ten and y from minus two to two. [She plots the graph of the function on the calculator]. If I find the maximum and the minimum, that will tell me how high and low a can be. Ok, so I get that maximum is 1.4 and the minimum is minus 1.4."
Author: "Have you now found *a*?"
Anna: "Yes."

Interpretation

It is clear from the transcript that when Anna is given the prompt, there are a two features of the problem that confuse her. The first is, as she mentions, the two unknowns or variables. There is both an *x* and an *a* in the equation. It seems obvious that she is not used to working with a single equation with more than one variable. She expresses her frustration as she doesn't know what to do or how to solve an equation with two variables. The second frustration, is seen when she asks what she is supposed to find. It seems she is uncertain of the properties of *a*. This is also seen when she asks if *a* is a number, an interval or an expression. On a more general basis, the frustration stems from not knowing how to solve the equation and not knowing what the solution is supposed to look like. However, with a little guidance from the author, she correctly answers that *a* is an interval between one and minus one.

The uncertainty regarding *a* is also seen when she tries to generalize her findings in the simpler equation to the original equation. Even though she found that *a* had to be between one and minus one in the simpler equation, she asks the author if a is supposed to be an interval, an integer or an expression. However, after the author asks what she found out when given the prompt, she correctly concludes that a must be an interval in the original equation as well. She then says that the interval in the original equation is also one to minus one, but she quickly corrects herself. First saying that the interval must be from two to minus two. Then, referencing the unit circle, she goes back to her original answer and says the interval is one to minus one. She doesn't offer any justification for chosing a 40 degree angle as an example or why she think calculating one example could verify her solution. It may be possible that by looking at the unit circle, she, explicitly or implicitly, concluded that the sum sine x plus cosine x would be greater near a 45 degree angle. When she sees that the answer is greater than one, but less than two, it seems she understands that the upper limit of the interval is somewhere between one and two. As Alf, Anna was now able to treat the equation as a function. Where *a* was a function of *x*. The reasoning structure that enable her to solve the equation is as following:

1. Strategy choice: The equation is also a function. The function can be plotted on the calculator. Finding the maximum and the minimum of the function will give the interval of *a*.
2. Strategy implementation: Plot $y = \sin x + \cos x$ on the calculator. The maxima is 1.4 and the minima is -1.4.
3. Conclusion: $a = [-1.4,\ 1.4]$

Although Anna needed some help from the author to solve the equation, there are some indications of creativity in her reasoning process. As Alf, her reasoning is flexible as she is able to view the equation as a function and find the interval by calculating the maxima and minima of the function. Here, two connections were necessary. First, she had to look at a as a function of *x* and consequently plot the left side of the equation on her calculator. Second, she needed to make the connection between finding a and finding the maxima and minima of the function. The reasoning sequence is flexible, it is based on mathematical properties and plausible. The author can not say with certainty that the reasoning sequence is new to Anna, but later in the interview she expresses that looking at the equation as a function was unusual; especially since *a* was a function of *x* and not *y* as a function of *x*. This might indicate that the reasoning sequence is indeed new, or rediscovered, to Anna.

As such, the reasoning sequence does fulfil all the criteria of creative reasoning, but at the same time she experienced several difficulties and needed help from the author to solve the equation.

– **Hege**

Description

Hege's reaction and work when given the prompt was very similar to what Anna did. Therefore, only a quick summary of her work will be given here. Hege needed some help from the author identifying the interval of a in the prompt. When she tried to generalize the results from the prompt and back to the original equation, she first expressed that the interval of *a* was from one to minus one in the original equation as well. She then reconsidered and concluded that the interval was from two to minus two. However, she quickly corrected herself, saying that cosine and sine couldn't both be one simultaneously. Hege quickly concluded that the interval of *a* had to be between one and minus one. Trying to verify her conclusion, she chose $x = \frac{\pi}{3}$ and found that a = 1.36 Afterwards, she said that the upper limit of *a* had to be greater than one, but less than two. She then became quiet for some time and the author asked her what she was thinking:

Hege: I'm trying to think. If I... [quiet for 10 seconds]. I have to maximize this. If I insert two pi. No, that's not right. No, I don't know how to do it. [Quiet for 15 seconds].
Author: Can you do it graphically?
Hege: Yes, you can. But you will find *y* instead of *a*.

She then proceeds to plot the graph of $y = \sin x + \cos x$ on her calculator. She finds the maxima and minima of the function.

Hege: The maximum and minimum of the graph is...eh...1.4 and minus 1.4. [quiet for 15 seconds]
Author: What does that tell you about *a*?
Hege: Doesn't it say *a* is between minus 1.4 and 1.4?
Author: Is that your answer?
Hege: I don't know when the equation is solved!

Interpretation

As Anna, Hege solved the equation by looking at I as *a* function, plotting the graph and finding the maxima and minima of the function using her calculator. The reasoning sequence is therefore:

1. Strategy choice: The equation is also a function. The function can be plotted on the calculator. Finding the maximum and the minimum of the function will give the interval of *a*.

2. Strategy implementation: Plots $y = \sin x + \cos x$ on the calculator. Find the maxima and minima of the function by using the tools available on the graphical calculator. The maxima is 1.4 and the minima is -1.4.
3. Conclusion: $a = [-1.4, 1.4]$

Like Anna, Hege realized that in order to find the interval of *a*, she had to, first, find the maximum value of the left side of the equation. However, unlike Anna, she did not view the equation also as a function. Only when the author asked her if she could find the maxima graphically, did she make the necessary internal connection. She was now able to treat the equation as a function. This is seen when she says that solving the problem graphically will give her the values for *y* and not *a*. It is reasonable to conclude that Hege's understanding of the function concept is not flexible. So the well known procedures and algorithms for finding the maxima and minima of a function are not applicable in this situation, where the right side of the equal sign is *a* and not *y*. Hege was not able to solve the equation without significant help from the author and although she found the interval of *y* of the function $y = \sin x + \cos x$, she was not immediately able to transfer this information to the case of *a*.

DISCUSSION

The purpose of the study was to investigate the mathematical reasoning of high achieving students in upper secondary school. Two research questions were formulated in the introduction and in this section, the author will try to answer both. When the students were first given the equation, all three attempted algorithmic reasoning. The students in varying ways attempted to find an algorithm or formula that would solve the equation. Algorithmic reasoning, when applied correctly, can reduce the cognitive load of solving mathematical problems. However, in this case, all three students attempted to use or find algorithms and/or formulas that were not helpful for solving the equation. A plausible explanation is that the students did not consider the intrinsic properties of the equation, but focused instead on the surface appearance. On the surface, the equation looked like equations they had met earlier in the textbook. The equations in the textbook could be solved using the formulas and algorithms Anna and Alf utilized, while Hege presumably looked for a similar algorithm or formula. This is seen in Hege's statements, where she said that she didn't remember how to solve the equation. The answer to the first research question, is therefore that in this case the high achieving students did display superficial reasoning when given an unfamiliar trigonometric equation.

The students' behaviour when they first approached the equation, also reveal that imitative reasoning is a strong characteristic of their mathematical reasoning. All three students' first strategy choice was to somehow simplify the equation using some standardized procedure or formula. Based on the students' behaviour, it became clear quite early on that they were not able to solve the equation on their own. After Alf was given the prompt, he quickly realized *a* had to be an interval in both the easier equation and in the original equation. He then solved the equation by looking at it as a function. Anna and Hege also solved the equation by looking it as a function, but unlike Alf they needed significant help from the author during the interview to make the necessary connection. However, regardless of the guidance from the author during the interview, all three students were able to view the equation as a function. This allowed them to solve the equation using simple calculations. The problem was that they did not make this connection without explicit or implicit guidance. Alf needed the prompt and Anna and Hege needed explicit guidance from the author during the interview. The ability to view the equation in multiple ways indicated versatility, but not necessarily adaptability (Sfard & Linchevski, 1994) as the students needed help to view the equation as a function.

Based on the observations made in this study, it is the author's claim that the students possess the necessary domain knowledge to solve the equation. The students' were, for all intents and purposes, able to make the necessary connections and calculations to solve the equation on their own. The problem was a more general and structural behavioural pattern. When the students first began working on the equation, they immediately began looking for a formula, algorithm or procedure that would let them solve the equation. Later in the interview, all three students were able to focus on the intrinsic properties of the equation and solve it, but only with explicit or implicit help from the author. As defined earlier in the article, the students' behaviour, arguments and written product is the basis for their way of understanding. By looking at the students' way of understanding, it may be possible to say something about their way of thinking. Based on the observations in this study, the author contends that it is plausible to suggest that the students' way of thinking vis-à-vis mathematical reasoning is characterised by an expectation that mathematical problems can be solved using a familiar procedure or algorithm.

The findings of this study reinforce earlier findings, which have indicated that even high achieving students display superficial reasoning when faced with a mathematical problem (Selden et al., 1994, Lithner, 2000 & Schoenfeld, 1985). Although the results are not generalizeable to all high achieving students, the three cases presented in this study do generate other questions. In particular, why do the students in this study display superficial reasoning? Why isn't their way of thinking, when given an unfamiliar equation, more flexible and creative? Hiebert (2003) argues that students learn what they are given an opportunity to learn. A possible explanatory hypothesis, is that the three students in this study have gotten good grades in school mathematics by focusing on memorizing and applying algorithms and procedures. They are simply a result of their learning milieu, which rewards imitative reasoning and not creative reasoning. However, the framework and methods used in this study can not give an answer to these questions. Future studies in which high achieving students' learning milieu is investigated is needed.

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