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Arctic Tipping Points

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MAT-3900 Master's Thesis in Mathematics, May 2015

Abstract

The Arctic is warming much faster than the entire planet, and this causes severe melting of sea ice. However, the climate of different regions of the Earth is interconnected, and changes in the amount of ice in the Arctic can dramatically affect the climate across the whole planet. Some scientists claim that a possible tipping point is the event of the ice-free Arctic Ocean in summer. Certain predictions point towards ice-free Arctic summers around the year 2050, whereas others predict this will occur in 2016. There are also others arguing that only a year-round sea ice loss can represent a tipping point in the Arctic. The disagreement between scientists on this topic is an indication that more detailed studies of tipping points are needed to be done.

In this thesis, we use five different models to explore possible tipping points of sea ice loss in the Arctic Ocean. The results show that the tipping point will most probably occur between years 2017 and 2021, with sea ice loss average from 11 million km sq to 4 million km sq during one seasonal cycle.

Acknowledgment

First of all, I would like to thank my supervisor Martin Rypdal for offering a very interesting project topic and for help in writing this thesis. I would also like to thank all my professors at the UiT The Arctic University of Norway and especially, professor Kristoffer Rypdal. This work could not have been done without his course of lectures and practical lessons in Climate Dynamics. I am also grateful to my parents, friends from the university and my boyfriend for support during this year.

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Chapter 1

Introduction

1.1 Background

The term "tipping point" is in widespread use in English. It is generally understood as a rapid and irreversible change of a system's state. Malcolm Gladwell, a journalist who wrote a book titled "The Tipping Point," relies on the following definition: "the tipping point is that one dramatic moment in an epidemic, when everything can change all at once" [Gladwell].

The term is used in many studies. In physics, for example, a tipping point is the point at which an object is no longer balanced and adding a small amount of weight can cause its drop. It is also very common to find this term being used in such sciences like sociology, catastrophe theory, economics and climatology. A closer look on the definition of a tipping point in climatology study shows that a very good example to explain tipping point is a transition of water into ice or vapor. So long as temperature of water stays above 0 and less than 100 degrees Celsius it remains liquid. But once it crosses any of those limits it changes its state: below 0, it becomes an ice and, when it reaches 100, it becomes a vapor [Ditlevsen and Johnsen].

In mathematics, the linearization of systems that are approaching bifurcation in their equilibrium solutions goes to zero tending to infinitely slow recovery from perturbations. This phenomenon is well known in dynamical systems theory as "critical slowing down," but has only recently been applied to climate dynamics [Lenton, 2011].

1.2 Significance of the project

Most climate scientists agree with climate warming trends over the last century. Figure 1.1 shows the data records from four major climate centers and as clearly seen, warming in the past few decades was rapid and the last decade has been the warmest on record. And even faster and more severe warming is going in the

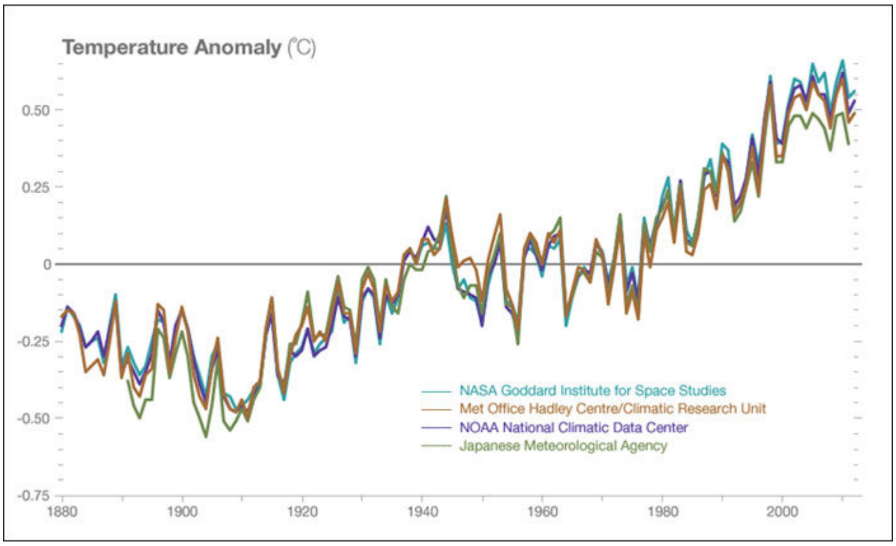


Figure 1.1: Temperature data records for the period 1880-2015. Courtesy of NASA Earth Observatory [Voiland, 2015].

Arctic. Many reports say that Arctic average temperature has risen at a rate of almost twice that of global average in the past few decades [Hassol]. It causes rapid and severe ice melting in the Arctic Ocean. Mostly, these changes will affect the Arctic region, however, the Earth's climate is an united system and changes in the amount of ice in the Arctic region can dramatically affect the whole climate across the planet. The typical Arctic climate processes have a significant impact on the global and regional climate. In addition, the Arctic provides important natural resources (such as oil, gas and fish) for the rest of the world. Melting of Arctic ice is also one of the factors contributing to the world ocean level increase [Hassol].

However, there is still no clear systematic work that has been done in order to predict possible future tipping points of sea ice melt in the Arctic region.

1.3 Objectives

During the project we will try to answer the following questions:

1. What are the possible tipping points in the Arctic?
2. How soon can they happen?
3. Are there true early warning signals for each of the found tipping points?
4. Which model works best in predicting the future possible tipping points of sea ice loss in the Arctic?

1.4 Structure of the thesis

Chapter 2 introduces the basic definitions and classifications of bifurcation theory for first and higher-dimensional systems. Continued, the theory about stochastic calculus is given in chapter 3. Such terms as variance, different kind of noises, the Langevin equation and the Ornstein-Uhlenbeck process are reviewed there. Chapters 2 and 3 were written and inspired by lecture notes from the course "Climate Dynamics" [Rypdal, K., 2014], lecture notes on stochastic processes with scaling properties [Rypdal, M., 2010] and the book "Nonlinear Climate Dynamics" [Dijkstra]. In chapter 4, the Arctic region, its climate system, specifically, albedo feedback, greenhouse gasses effect on climate change, and Earth's energy balance equations are described. Also this chapter gives a review of the past and future possible Arctic tipping points as seen by publishing climate scientists. Chapter 6 gives some introduction to modeling and early warning signals of tipping points. The prediction of sea ice loss tipping points for the next 10 years in the Arctic is given in chapters 6 and 7. Results and comparison of these models are discussed in chapter 8.

Chapter 2

Bifurcations in dynamical systems

In the following chapter, we will give definitions to fixed point, phase portrait, vector field, equilibrium and we will also define and classify bifurcations in first- and higher-dimensional dynamical systems, we will give descriptions of saddle-node, transcritical, pitchfork and Hopf bifurcations.

2.1 Basic definitions

Let us introduce a general system:

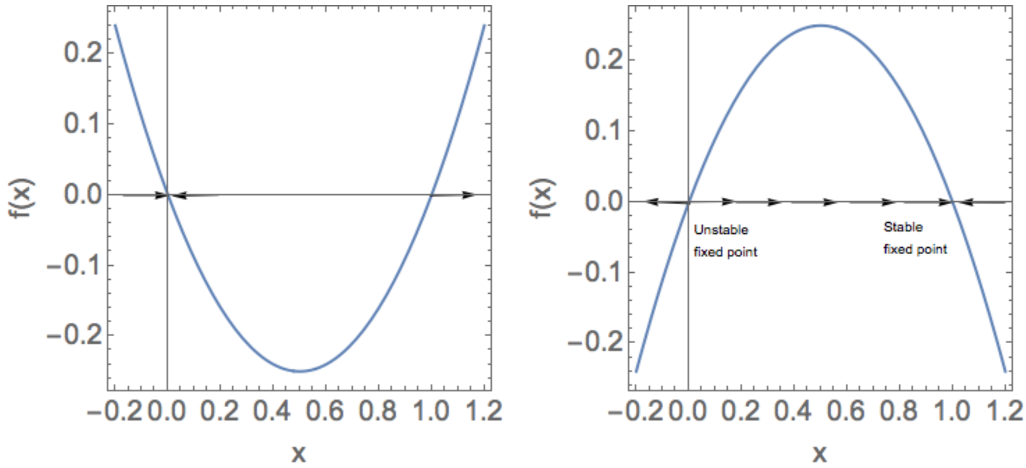
$$y_1 = f_1(x_1, \dots, x_n),$$

$$\dots$$

$$y_n = f_n(x_1, \dots, x_n).$$

This is a *n-dimensional* or *n-th order dynamical system.* We will consider its solutions as trajectories flowing through n-dimensional phase space with coordinates $(x_1, \dots, x_n)$. For example, we consider one-dimensional dynamical system $y = f(x)$. A sketch of the vector field on the graph of $f(x)$ is shown in Figure 2.1(a).

To find fixed points we use the equation $f(x^*) = 0$; sometimes such points are called “equilibrium”. Equilibrium is stable if all sufficiently small disturbances away from it damp out in time, and unstable if disturbances grow in time.



(a) (b)

Figure 2.1: Example of vector field (a). Phase portrait of $f(x) = x(1-x)$ (b).

In order to show the algorithm of finding stability of fixed points in dynamical systems, we will consider the equation $y = x(1-x)$; we will find fixed points and check their stability.

We have $f(x) = x(1-x)$. To find the fixed points, we set $f(x^*) = 0$ and solve for x^*. Thus, $x_1^* = 0$ and $x_2^* = 1$ are the fixed points. To determine their stability, we plot $x(1-x)$ and then sketch the vector field as shown in Figure 2.1(b).

2.2 First-order bifurcations

The mechanism, by which fixed points are created and destroyed or their stability changes, is called *bifurcation*, and the parameter value, at which they occur, is called *a bifurcation point*.

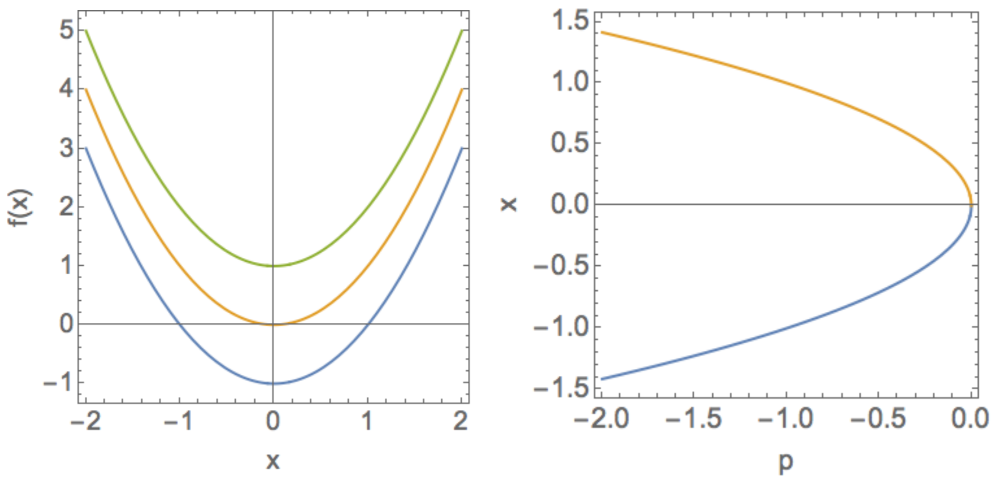
There are different types of bifurcations. In the following sections they are defined.

2.2.1 Saddle-node bifurcation

Saddle-node bifurcation occurs when two fixed points come into existence as a parameter reaches a critical value. A basic example of this bifurcation is given by a first order system:

$$d_t x = f(x, p) = p + x^2,$$

where p is a parameter that can be positive, negative or zero. When $p < 0$, there is one stable and one unstable fixed point (Figure 2.2). As p approaches 0, two fixed points become one half-stable fixed point that vanishes as soon as p becomes positive. Then, we can say that the bifurcation occurs at $p = 0$, when the system with two fixed points turns into the system with no fixed points.



(a) (b)

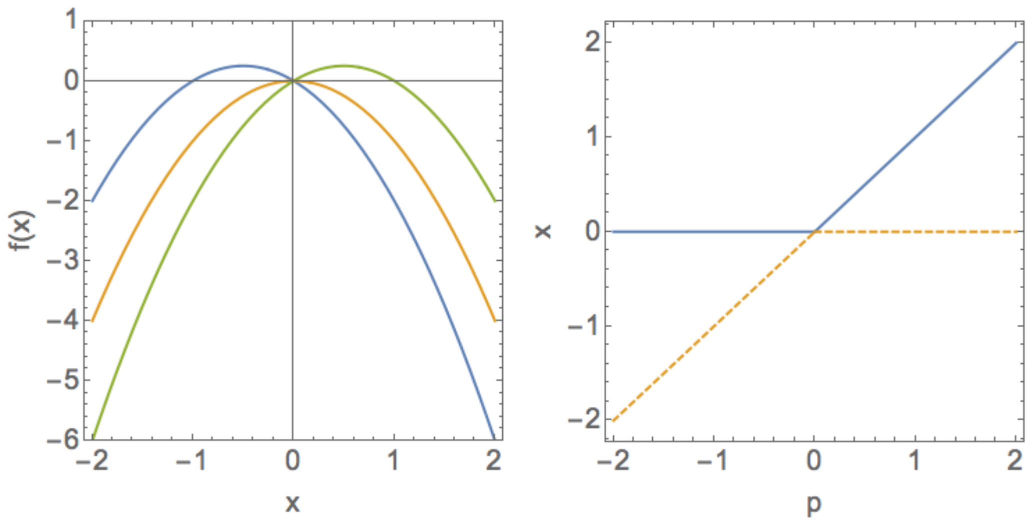
Figure 2.2: Velocity fields (a) and bifurcation diagram (b) of saddle-node bifurcation.

2.2.2 Transcritical bifurcation

Transcritical bifurcation occurs, when two fixed points exchange stability properties as a parameter reaches a critical value. The normal form of this bifurcation is given by the following system:

$$d_t x = f(x, p) = px - x^2.$$

For $p < 0$, there is one unstable fixed point at $x^* = p$ and one stable fixed point at



(a) (b)

Figure 2.3: Velocity fields (a) and bifurcation diagram (b) of transcritical bifurcation.

$x^* = 0$. When $p = 0$, there is only one fixed point in origin. Finally, when $p > 0$, the origin is unstable and $x^* = p$ is now stable (Figure 2.3). The bold lines on the transcritical bifurcation diagram show the stable fixed points and the dashed (dotted) lines show the unstable fixed point.

2.2.3 Pitchfork bifurcation

There are two types of *Pitchfork bifurcation*: supercritical and subcritical. When a stable point splits into one unstable and two stable, it is *supercritical pitchfork bifurcation*, and if a stable point splits into one stable and two unstable, it is *subcritical pitchfork bifurcation.*

The normal form of supercritical pitchfork bifurcation is $d_t x = f(x, p) = px - x^3$, and subcritical: $d_t x = f(x, p) = px + x^3$.

The fixed points are $x = 0$ and $x = \pm\sqrt{r}$ for $p > 0$ (supercritical, Figures 2.4(a) and 2.4(b)) and $x = \pm\sqrt{(-p)}$ for $p < 0$ (subcritical, Figures 2.4(c) and 2.4(d)).

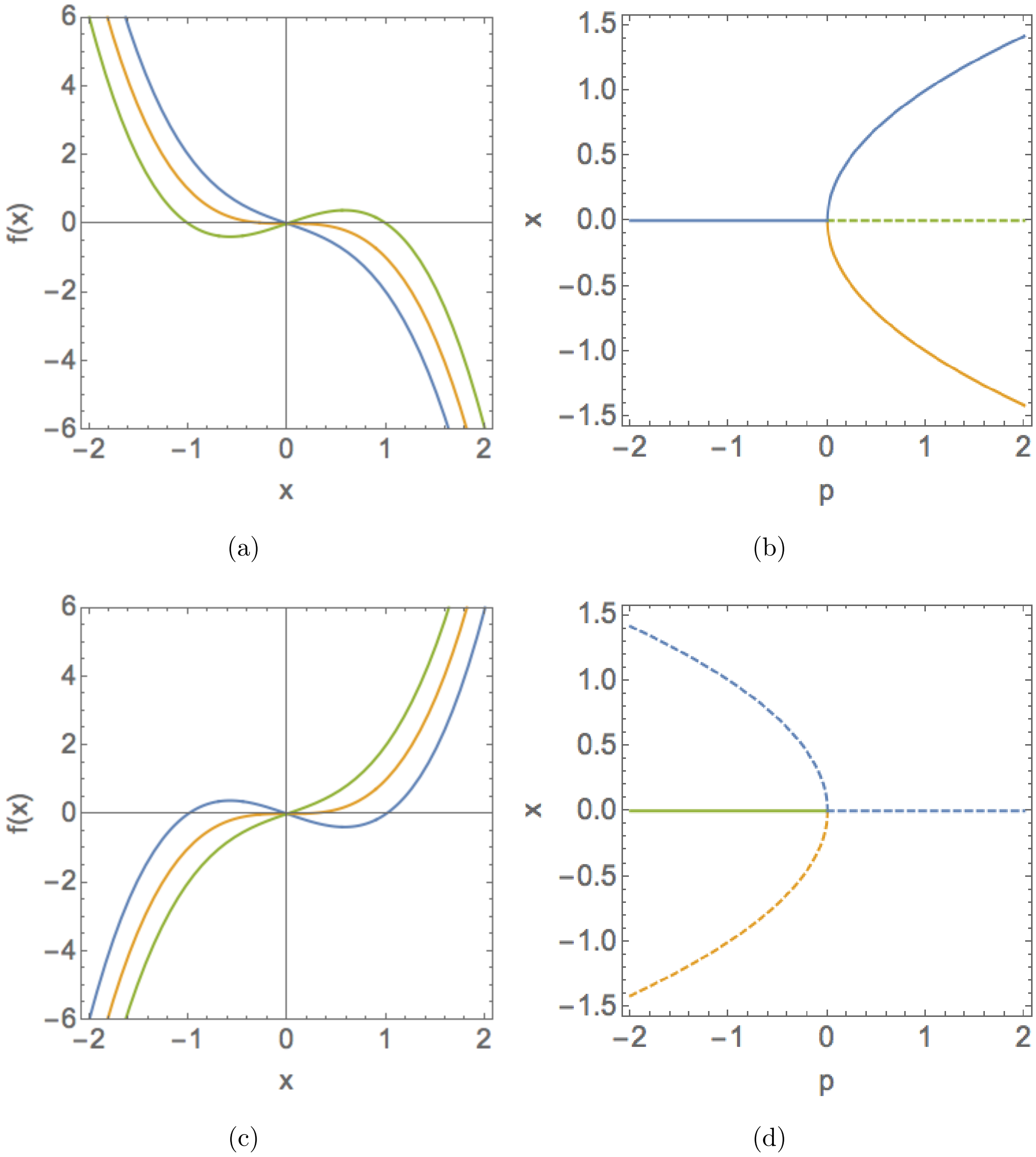


Figure 2.4: Velocity fields (a) and bifurcation diagram (b) of supercritical pitchfork bifurcation. Velocity fields (c) and bifurcation diagram (d) of subcritical pitchfork bifurcation.

2.3 Bifurcations in higher-dimensional systems

Previously discussed fixed points have analogical classifications in 2–D and higher dimensions. Nothing new happens when more dimensions are added — all the action is confined to a 1–D subspace, along which a bifurcation occurs, while in the extra dimensions the flow can be either attraction or repulsion from that subspace [Strogatz].

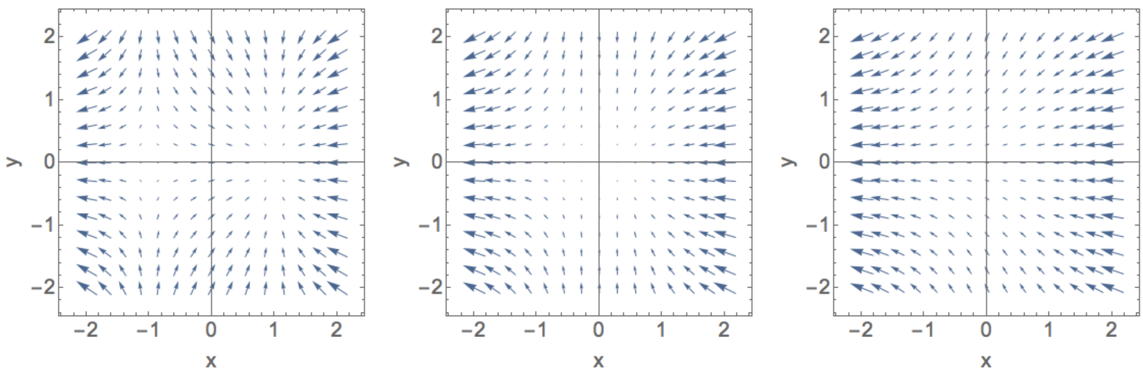
2.3.1 Two-dimentional saddle-node bifurcation

As an example of *two-dimensional saddle-node bifurcation*, which is the basic mechanism for the creation and destruction of fixed points, we consider the following system:

$$d_t x = p - x^2, \;\; d_t y = -y.$$

In the x-directions, we can see the usual 1–D saddle-node bifurcation, while in y-direction, the motion is exponentially damped.

In Figure 2.5(a), when $p > 0$, there are two fixed points: saddle-node at $(x^*, y^*) = (\sqrt{p}, 0)$ and saddle at $(\sqrt{-p}, 0)$. Then in Figure 2.5(b), there is a fixed point at $x = 0$ for $p = 0$, and finally, in Figure 2.5(c), there is no real fixed points for the case when $p < 0$.



(a) (b) (c)

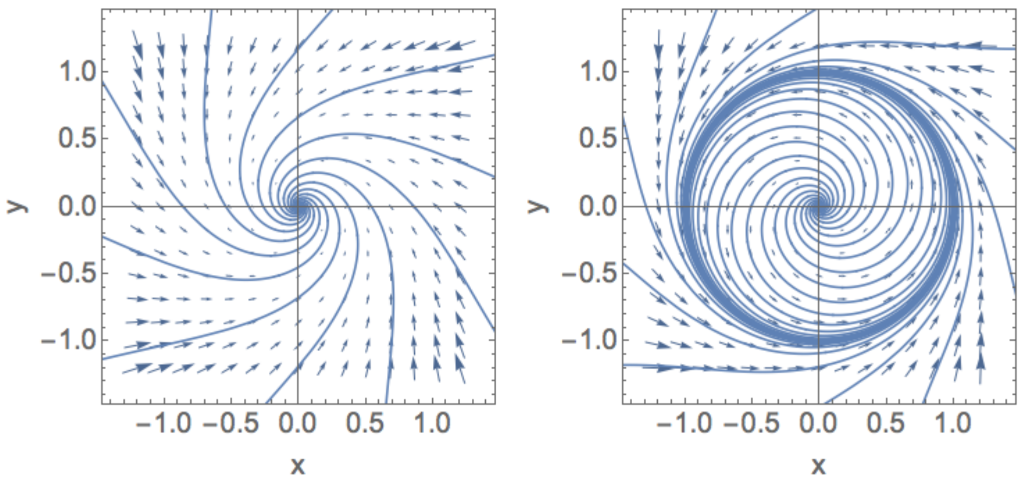
Figure 2.5: Phase portrait of two-dimensional saddle-node bifurcation. First stage (a), second stage (b) and final stage(c).

2.3.2 Hopf bifurcation

Let us consider a system in polar coordinates:

$$d_t p = pr - p^3, \; d_t\theta = \omega.$$

This system has cyclic (oscillating) solutions. They are illustrated in Figure 2.6.



(a) (b)

Figure 2.6: Phase portrait of hopf bifurcation for $\omega = 1$. One stable spiral node for case when $p = -1$ (a) and stable limit cycle and unstable origin for case when $p = 1$ (b).

Figure 2.6(a) shows the situation when $p < 0$ and there is just one stable spiral node at the origin. In Figure 2.6(b), we see a solution $r = \sqrt{p}$, $\theta = \omega t$ and it is a limit cycle that is stable because it attracts orbits from inside and outside the circle. In this case, the origin is unstable. In Cartesian coordinates, we have

$$x = r\cos\theta, \; y = r\sin\theta,$$

then, the system will take the following form:

$$d_t x = [p - (x^2 + y^2)]x - \omega y, \; d_t y = [p - (x^2 + y^2)]y + \omega x.$$

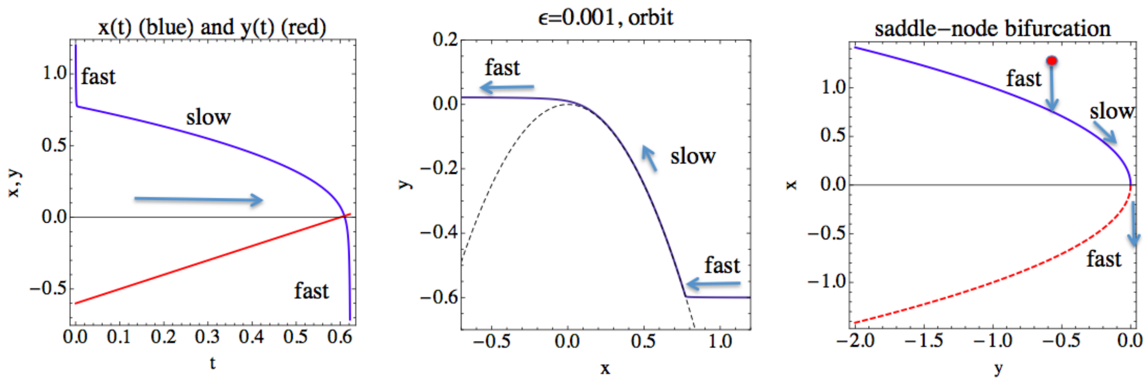
Hopf bifurcation occurs when a stable spiral node in origin splits into an unstable spiral node in origin and a stable limit cycle.

2.3.3 Fast-slow systems

There is given a system:

$$\varepsilon d_t x = -y - x^2, \; d_t y = 1, \tag{2.1}$$

with the initial condition $(1.2, \; -0.6)$. Here, we have $\varepsilon << 1$. Figure 2.7(a) shows how the slow variable grows slowly and linearly with time, while $x(t)$ relaxes fast because of the small ε to the stable equilibrium given by the equation $-y - x^2 = 0$. The system then moves slowly along this branch until bifurcation takes place at $y = 0$. After this stability is lost and $x(t)$ goes fast to $-\infty$.



(a) (b) (c)

Figure 2.7: Fast-slow system. Velocity field of the system (a), case when $\varepsilon = 0.001$ (b) and saddle-node bifurcation diagram (c).

We will use these kind of systems in chapter 3 for showing an increase of variance prior bifurcation.

Chapter 3

Stochastic dynamical systems

In the following chapter, we will try to describe some basic definitions and theorems of stochastic calculus, as well as take a closer look on general stochastic differential equations and their solutions. A small classification of noises depending on the Hurst exponent will be given and we will also define the Langevin equation and the Ornstein-Uhlenbeck process for stochastic dynamical systems. We will show that an increase of variance in fast-slow systems takes place before bifurcation.

3.1 Stochastic calculus

Definition 3.1. *A stochastic process* $(X_t, t \in T) = \big(X_t(\omega), t \in T, \omega \in \Omega\big)$ is a time series of random variables X_t, defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and where T is a time interval.

If we take a fixed time, then $X_t(\omega)$ is just a random variable. If we take some particular ω, then we get just one sample path of the process — a function of time. The following properties can be defined for any stochastic processes:

- The expectation function is equal to $\mu_X(t) = \mathrm{E}[X_t]$,
- The covariance function is $c_X(t, s) = \mathrm{Cov}(X_t, X_s)$,
- The variance function is $\sigma_X(t) = c_X(t, t)$.

Definition 3.2. *A normal or Gaussian distribution*, $N(\mu, \sigma^2)$ of one single random variable is a distribution with the following PDF (probability density function):

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

A random variable with a Gaussian distribution is said to be normally distributed and is called a normal deviate.

If $\mu = 0$ and $\sigma = 1$, the distribution is called the standard normal distribution or the unit normal distribution denoted to $N(0, 1)$ and a random variable with that distribution is a standard normal deviate [Shumway and Stoffer].

Definition 3.3. *A Gaussian white-noise* can be defined in $T = [0, 1]$ the following way: n is a random positive integer, so that $0 \leq t_1 \leq \ldots \leq t_n \leq 1$, then $X_{t_1}, \ldots, X_{t_n}$ are independent and standard normal distributed, $N(0, 1)$: at any $t, s \in [0, 1]$, where $t \neq s$, we have the following mean value $\mu_X(t) = 0$, covariance $c_X(t, s) = 0$ and variance $\sigma^2{}_X(t) = 1$.

Definition 3.4. *A Weiner process* W_t, $t \in [0, \infty]$, is a process with the following properties:

- $W_0 = 0$,
- W_t has stationary, independent increments, i.e. if $X_t - X_s \overset{\mathrm{d}}{=} X_{t+h} - X_{s+h}$ for all (t, s, h) — stationary increments, and if $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent for every $t_1 < t_2 < \ldots < t_n$ — independent increments,
- W_t has a $N(0, t)$ distribution for all t, i.e. the PDF is Gaussian with $\mu_W(t) = 0$, and $\sigma^2_W(t) = t$,
- W_t has continuous sample paths.

3.1.1 The Hurst exponent and Brownian motion

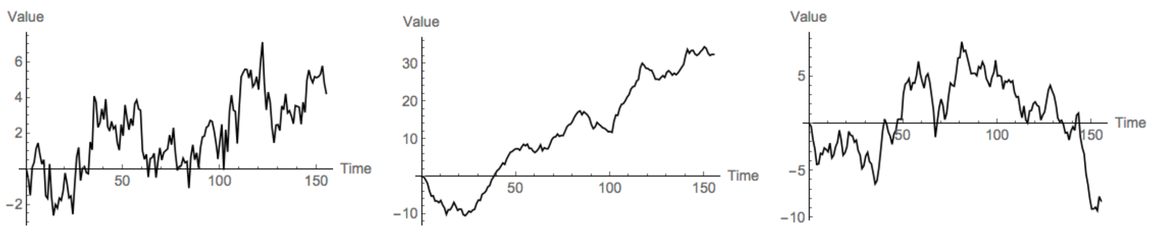
Definition 3.5. *The Hurst exponent (H)* is a measure of autocorrelation (persistence and long memory) and usually defined in terms of the asymptotic behavior of the rescaled range as a function of the time interval of a time series as follows:

$$\mathrm{E}\left[\frac{R(n)}{S(n)}\right] = Cn^H \text{ as } n \to \infty,$$

where:

- $R(n)$ is a range (the difference between the biggest and the smallest values) of the first n values, and $S(n)$ is their standard deviation (the difference of each data point from the mean in square),
- $\mathrm{E}[x]$, is the expected value,
- n is the time interval of the observation (number of data points in a time series),
- C is a constant.

The Hurst exponent is usually defined on the interval $(0, 1)$. Although, cases for $H = 0$ and $H = 1$ are also possible, they are not interesting for our further work.



(a) (b) (c)

Figure 3.1: Fractional Brownian motion realizations with $H = 0.25$ (a), $H = 0.75$ (b) and $H = 0.5$ (c).

For $0 < H < 0.5$ (for example, in Figure 3.1(a)), a time series has negative autocorrelation (e.g. a decrease between values will probably be followed by another decrease). It is not interesting for us. The second case, when a value of the Hurst exponent H is between 0.5 and 1 (for example, Figure 3.1(b)), indicates a time series with positive autocorrelation (e.g. an increase between values will probably be followed by another increase). A value of $H = 0.5$ indicates a random walk (Figure 3.1(c)), where it is equal possibility to have a decrease or an increase after any particular value (e.g. the time series has no memory of previous values) [Qian and Khaled]. The last two cases are the most significant for our further analysis in chapters 5, 6 and 7.

Definition 3.6. Let $H \in (0,1)$. *A fractional Brownian motion* with the Hurst exponent H is a Gaussian process with:

1. $\mathrm{E}[X(t)] = 0$ for all t,
2. $\mathrm{E}[X(t)X(s)] = \frac{1}{2}\,\mathrm{E}[X(1)]^2\big(t^{2H} - s^{2H} - |t-s|^{2H}\big)$ for all t, s.

A standard Brownian motion with $H = 1/2$ is sometimes called the Weiner process (random walk).

3.1.2 Stochastic integral and differential equation

Definition 3.7. *A stochastic integral* is a mean-square limit of a partial sum of stochastic differentials:

$$X \equiv \int_0^T h(t)dW(t) = \mathrm{ms} - \lim_{N\to\infty} X_N \equiv \mathrm{ms} - \lim_{N\to\infty} \sum_{j=0}^{N-1} h(\tau_j)(W_{t_{j+1}} - W_{t_j}). \quad (3.1)$$

Mean-square equality means:

$$X \overset{\mathrm{ms}}{=} Y \Leftrightarrow \mathrm{E}[(X-Y)^2] = 0,$$

and the mean-square limit equals to:

$$X = \mathrm{ms} - \lim_{N\to\infty} X_N \Leftrightarrow \lim_{N\to\infty} \mathrm{E}[(X - X_N)^2] = 0.$$

Definition 3.8. *The Itô integral* is defined as the following mean-square limit:

$$\int_0^T W_t dW_t = \mathrm{ms} - \lim_{N\to\infty} (\frac{1}{2}W_T - \frac{1}{2}\sum_{j=0}^{N-1} \triangle {W_j}^2) = \frac{1}{2}({W_T}^2 - T). \quad (3.2)$$

Theorem 3.1. *Itô's third lemma:* Let $f(t, X_t)$ be a stochastic process, where $f : R^2 \to R$ is a smooth function and X_t is given in the following form:

$$X_t = X_0 + \int_0^t A^1(s, X_s)ds + \int_0^t A^2(s, X_s)dW_s,$$

where $A^1(s)$ and $A^2(s, W_s)$ are smooth functions. Therefore, the difference is true:

$$f(t, X_t) - f(s, X_s) =$$
$$\int_s^t \left[\frac{\partial f}{\partial x(x, X_x)} + A^1(x, W_x)\frac{\partial f}{\partial W_x(x, X_x)} + \frac{1}{2}\big(A^2(x, W_x)\big)^2 \frac{\partial^2 f}{\partial W_x^2(x, X_x)}\right]dx +$$
$$\int_s^t A^2(x, W_x)\frac{\partial f}{\partial W_x(x, X_x)}dW_x.$$

Definition 3.9. *A general stochastic differential equation* is an equation:

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t, \tag{3.3}$$

where the functions $a(t, x)$ and $b(t, x)$ are smooth in both variables t and x.

If we take the Itô integral of (3.3), we will get a solution:

$$X_t - X_0 = \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s. \tag{3.4}$$

Definition 3.10. Let X_t be a solution of the SDE (3.3). If it has the following properties:

- X_t satisfies (3.4), and at some time t, it depends on a sample path of Weiner process W_s (where $s < t$) and on functions $a(t)$ and $b(t)$,

- the integrals in the equation (3.4) are well-defined: the first integral on the right part of the equation is a Riemann integral and the second one is the Itô stochastic integral,

then such solution is called *a strong solution.*

The SDE (3.3) has an unique strong solution on the time interval $[0, T]$, if the functions $a(t, x)$ and $b(t, x)$ are point-wise continuous in both variables and in addition they are Lipchitz continuous in the second variable, i.e. these functions are limited in how fast they can change in order to second variable.

In case when $a = 0$ and $b = 1$, a solution of the Itô integral is just a Weiner process.

Definition 3.11. If $a(t, x)$ and $b(t, x)$ are linear, i.e. $a(t, x) = c_1(t)x + c_2(t)$, $b(t, x) = \sigma_1(t)x + \sigma_2(t)$, then the SDE (3.3) is *linear*.

In case when $\sigma_1 = 0$, the equation (3.4) can be written in the following way:

$$X_t - X_0 = \int_0^t \Big[c_1(s)X_s + c_2(s) \Big] ds + \int_0^t \sigma_2(s) dW_s. \tag{3.5}$$

We can use Itô's third Lemma (Theorem 3.1) in order to find solution to (3.5). We introduce the linear function $f(t, X_t) = \alpha(t)X_t$, where $\alpha(t) = e^{-\int_0^t c_1(s)ds}$, and in this case, smooth functions are the following:

$$A^1 = c_1 X + c_2, \; A^2 = \sigma^2,$$

then the equation (3.1) takes the form:

$$X_t = \frac{1}{\alpha(t)} \Big[X_0 + \int_0^t \alpha(x)c_2(x)dx + \int_0^t \alpha(x)\sigma_2(x)dW_x \Big]. \tag{3.6}$$

3.1.3 The Langevin equation and the Ornstein-Uhlenbeck process

Definition 3.12. A special case of (3.3), when the constants are:

$$c_1(t) = -\gamma, \; c_2 = 0, \; \omega_2 = \sigma,$$

gives *the Langevin equation*:

$$dX_t = -\gamma X_t + \sigma dW_t. \tag{3.7}$$

Therefore, we get $\alpha(t) = e^{\gamma t,}$ and the solution (3.6) reduces to the process:

$$X_t = e^{\gamma t}\big[X_0 + \sigma \int_0^t e^{\gamma s} dW_s\big]. \tag{3.8}$$

Definition 3.13. The process (3.8) is called *the Ornstein-Uhlenbeck process.*

This process describes Brownian motion of particles immersed in one-dimensional fluid. Mean value of this process is equal to:

$$\mu_X = e^{-\gamma t}\,\mathrm{E}[X_0] = \mu_X(0)e^{-\gamma t}$$

and variance is:

$$\mathrm{Var}[X_t] = e^{-2\gamma t}\left(\mathrm{Var}[X_0] + \sigma^2 \frac{e^{2\gamma t} - 1}{2\gamma}\right) = \frac{\sigma^2}{2\gamma}(1 - e^{2\gamma t}).$$

The autocorrelation function of the OU-process is:

$$\mathrm{Cor}[X_t, X_{t+k}] = \frac{e^{-\gamma k}(1 - e^{-2\gamma t})}{\sqrt{(1 - e^{-2\gamma t})(1 - e^{-2\gamma (t+k)})}}.$$

When $t \to \infty$ correlation goes to:

$$\lim_{t\to\infty} \mathrm{Cor}[X_t, X_{t+k}] = e^{-\gamma k}.$$

3.2 Stochastic dynamical systems and their solutions

Definition 3.14. *A stochastic dynamical system* is a mapping $\Theta(t)$ from direct product of a sample space ω (containing samples ω of a driving Wiener process) and a state space X (containing state vectors x) in themselves:

$$\Theta(t) : \Omega \times X \to \Omega \times X,\ (\omega, x) \to (\theta_t\omega,\ \phi(t, \omega)[x]),$$

where the driving system has the following properties:

1. $\theta_0\omega(r) = \omega(r)$,

2. $\theta_0 \circ \theta_s \omega(r) \equiv \theta_t\big[\omega(s+r) - \omega(s)\big] = \big[\omega(t+s+r) - \omega(t)\big] - \big[\omega(t+s) - \omega(t))\big] = \omega(t+s+r) - \omega(t+s) = \theta_{t+s}\omega(r)$.

3.2.1 Pullback attractor

Let us consider an example:

$$dX_t = \left[(\alpha + \frac{\sigma^2}{2})X_t - \beta {X_t}^3\right]dt + \sigma X_t dW_t, \tag{3.9}$$

where W_t is a sample path.

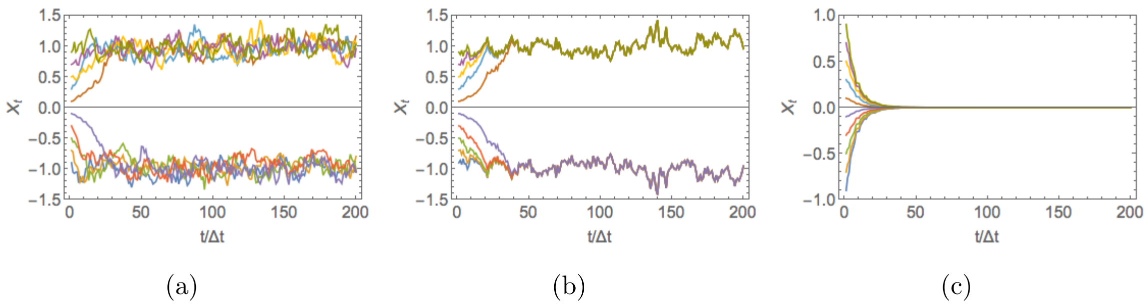


Figure 3.2: Numerical solution for the equation 3.9 for 10 different initial conditions and a new sample path for every run (a), one sample path for every run for $\alpha < 0$ (b) and $\alpha > 0$ (c).

Figure 3.2(a) shows a numerical solution for the equation (3.9) for 10 different initial conditions and for every run, we take a new sample path W_t. In Figures 3.2(b) and 3.2(c), we can observe that if we take one and the same sample path W_t and 10 different conditions for large t, initial conditions are forgotten and the solution converges to *pullback attractor* (fixed point for deterministic version), which is determined by the Weiner sample path. The case for $\alpha < 0$ is shown in Figure 3.2(b) and $\alpha > 0$ in Figure 3.2(c).

3.2.2 Variance in fast-slow systems

Definition 3.15. An equation of the following form:

$$\frac{\partial p}{\partial t} + \frac{\partial (ap)}{\partial x} - \frac{1}{2}\frac{\partial^2 (pb^2)}{\partial x^2} = 0 \tag{3.10}$$

is called *the forward Fokker-Plank equation.*

In case of the OU-process, we have $a(x,t) = -\gamma$ and $b(z,t) = \sigma$, then the forward Fokker-Plank equation for the OU-process reduces to:

$$\frac{\partial p}{\partial t} = \frac{\partial (\gamma x p)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2},$$

with the boundary conditions: $p, \frac{\partial p}{\partial x} \to 0$ as $x \to \infty$.

Let a system:

$$dX_t = f(X_t, Y_t)dt + \sigma dW_t,\ dY_t = \varepsilon dt$$

be a fast-slow system, where $\varepsilon << 1$. X_t is a fast stochastic process and Y_t is a slow one. Since ε is very small, Y_t can be considered as a constant ($Y_t \approx y$). The forward Fokker-Plank equation for the PDF $p(x,t;y)$ can be found from the equation (3.10) with $a = f(x,y),\ b = \sigma$:

$$\frac{\partial p}{\partial t} = -\frac{\partial (fp)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2}. \tag{3.11}$$

In order to find a stationary solution, we set the left side to zero and integrate over the interval (a,x) :

$$\bar{p}(x,y) = \frac{1}{N(y)} e^{\int_a^x \frac{2}{\sigma^2} f(s,y)ds}, \tag{3.12}$$

where $N(y)$ is a normalization factor determined by the following condition:

$$\int_{-\infty}^{\infty} \bar{p}(x,y)dx = 1.$$

The deterministic equation (2.1) $f(x,y) = -y - x^2$, which we have considered in chapter 2, gives two fixed points: stable at $x = \sqrt{-y}$ and unstable at $x = -\sqrt{-y}$;

this gives rise to saddle-node bifurcation at $y = 0$. For $y > 0$, there is no fixed points and $x \to -\infty$. For $y < 0$ and $x < -\sqrt{-y}$, we again have no fixed points and $x \to -\infty$. We set the boundary condition to be:

$$\bar{p}(x < -\sqrt{-y}) = 0.$$

Hence, we choose the lower integral limit for the equation (3.12) to be $a = -\sqrt{-y}$ and $\frac{2}{\sigma^2} = 1$for simplicity; then an integral takes the form:

$$\int_a^x (-y - s^2)ds = -yx - \frac{1}{3}x^3 + \frac{2}{3}(-y)^{3/2},$$

a normalization condition then becomes:

$$N(y) = \int_a^\infty e^{-yx-\frac{1}{3}x^3+\frac{2}{3}(-y)^{3/2}}dx,$$

and finally, the stationary PDF is:

$$\bar{p}(x,y) = \frac{e^{-yx-\frac{1}{3}x^3+\frac{2}{3}(-y)^{3/2}}}{\int_a^\infty e^{-yx-\frac{1}{3}x^3+\frac{2}{3}(-y)^{3/2}}dx} = \frac{e^{-yx-\frac{1}{3}x^3}}{\int_a^\infty e^{-yx-\frac{1}{3}x^3}dx}.$$

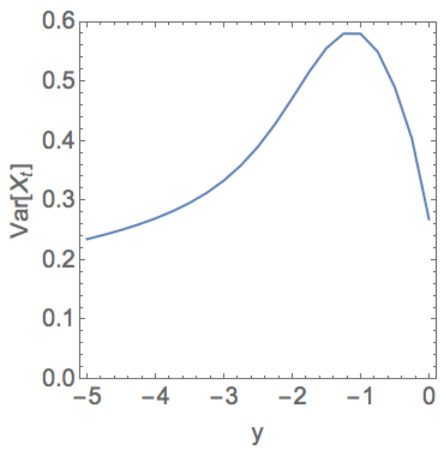


Figure 3.3: Variance of fast-slow system.

Variance of X_t is then equal to:

$$\mathrm{Var}[X_t](y) = \mathrm{E}[{X_t}^2] - \mathrm{E}[X_t]^2 = \int_a^\infty x^2 \bar{p}(x,y)dx - \left[\int_a^\infty x\bar{p}(x,y)dx\right]^2. \quad (3.13)$$

Putting the found PDF into the equation (3.13) gives an expression that can be computed with Mathematica (Figure 3.3). We can observe a big rise of variance not long before y approaches a bifurcation point at $y = 0$ and then a big drop just before the bifurcation point.

An increase of a fluctuation level as bifurcation is approached in stochastic dynamical systems occurs in other types of bifurcations, and is a very important indicator of early warning of a critical transition.

Chapter 4

Arctic climate system

In the following chapter, we will describe the Arctic region, we will also consider what is changing in the Arctic climate system and what impacts it causes. Past and possible future tipping points according to climate scientists will be presented in the rest of the following chapter.

4.1 The Arctic region

The Arctic is a northern polar area of the planet Earth surrounding the North Pole. It consists of the Arctic Ocean and parts of Alaska (United States), Canada, Finland, Greenland (Denmark), Iceland, Norway, Russia and Sweden lands [Smithson et al.]. Scientists use the definition of *the Arctic* by selecting the locations where average daily summer temperature does not rise above 10 degrees Celsius (50 degrees Fahrenheit). In the map to the right (Figure 4.1), the

Figure 4.1: Arctic region. The following map was produced by the U.S. Central Intelligence Agency [Cia.gov].

red curve shows such a definition [Cia.gov].
The area of the Arctic region is then 21 million sq km. However, sometimes the Arctic is considered as an area above Arctic Circle, which lies at $66°32''N$ (the dashed blue circle in Figure 4.1). Then the area is equal to 27 million sq km [Hollar].

Nature of the Arctic is unique. It is the habitat of a number of unique animals: muskoxen, reindeers and polar bears. During polar summer, seals, walrus and also several species of cetacean: beluga and killer whales, can be found in the seas of the Arctic.

The first representatives of Homo sapiens entered on the coast of the Arctic Ocean about 30 000 years ago. Presently, Arctic indigenous people preserve the traditional way of their ancestors' lives. Circumpolar North indigenous peoples include the Buryats, Chukchi, Evenks, Inupiat, Khanty, Koryaks, Nenets, Sami, Yukaghir, and Yupik, who still refer themselves to Eskimo. Nowadays together with indigenous people and settlers from the south, the Arctic population is about 400 thousand people [Hoffecker].

The Arctic's climate is characterized by cold winters and cool summers. Precipitation mostly comes in the form of snow. The average temperature of the coldest winter month — January — is ranging from $-2 \ldots -4$°C in the southern part of the Arctic region to -25°C in the north of the Barents Sea, in the Baffin, Chukchi and in the west of the Greenland seas and from $-32 \ldots -36$°C in the Siberian region, in the north of Canada; and in the part of the Arctic basin to $-45 \ldots -50$°C in the central part of the Greenland. Minimum temperature in these areas sometimes falls down to $-55 \ldots -60$°C [Serreze et al., 2007].

4.2 Energy budget

The weather on the Earth and the global climate are determined by *the Earth's energy budget* or *the Earth's radiation balance* that describes the net flow of energy into Earth in the form of shortwave radiation and outgoing infrared long-wave radiation into space. The Earth's energy budget is shown in Figure 4.2.

The main flow of energy in the atmosphere of the Earth is provided by solar radiation in the spectral range from 0.1 to 4 microns. The energy flux density

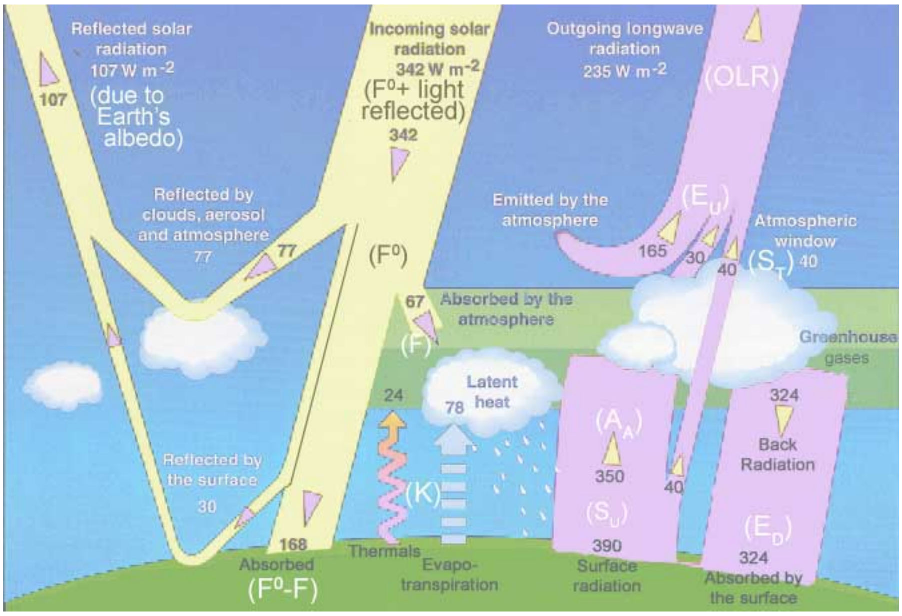


Figure 4.2: The Earth's Energy Balance, shown in W/m^2 [Kiehl and Trenberth].

from the Sun at a distance of 1 astronomical unit is about 1367 W/m^2 (solar constant). According to data for the period 2000–2004 averaged over time and over the surface of the Earth, this stream is 342 W/m^2 or 1.74×10^{17} W counting on the entire surface of the Earth [Trenberth et al.].

Out of the total amount of solar radiation reaching the Earth, approximately 30% is immediately reflected by the surface of the Earth due to albedo and clouds, and the rest is absorbed by the atmosphere and the Earth's surface. Absorption in the atmosphere is caused mainly by clouds and aerosols [Trenberth et al.].

30% of the energy absorbed by the Earth's surface goes back into space in the form of thermal radiation in the range of 3–45 microns and due to the evapotranspiration [Trenberth et al.].

At the same time, the Earth radiates 390 W/m^2, most of it is absorbed by the atmosphere, 90% of which comes back as the return of atmospheric radiation due to greenhouse gases feedback. 10% goes to space via an atmospheric window.

Thus, the total radiation absorbed by the surface of the Earth is 374 W/m^2 corresponding to average temperature 288 K (15°C) [Trenberth et al.].

For the Earth's temperature to be stable over long periods of time, incoming energy and outgoing energy have to be equal. In other words, the energy budget at the top of the atmosphere must be in a state of balance — *radiative equilibrium*. What can trigger the changes in such a balanced heat budget of the planet?

The trigger is human activities, primarily, the burning of fossil fuels (coal, oil and natural gas) and, secondarily, the forest decline leads to an increase in atmospheric carbon dioxide, methane and other heat-trapping (greenhouse) gases emissions. Since the industrial revolution, the concentration of carbon dioxide in the atmosphere has increased by 35%, while average global temperature has increased by 0.6°C. The international scientific community has agreed that the greater part of the observed warming over the last 50 years is due to an anthropogenic forcing [Stocker et al.].

Higher temperature leads to ice melt, therefore, at a warmer surface, solar radiation, which had been previously reflected back in space due to high albedo of ice and snow, is now being absorbed, causing further increased warming. As a consequence, even more ice melts [Houghton].

4.3 Arctic climate change and its impacts

Near-surface temperatures in the Arctic are rising two to four times faster than over the globe as a whole (Figure 4.3) [Screen et al.]. But why is the Arctic warming faster?

Firstly, as the Arctic snow and ice melt, darker land and ocean surfaces absorb more solar energy, increasing the warming of the Arctic (Figure 4.4). Secondly, most of the Arctic extra energy coming to the surface due to the increase in the concentration of greenhouse gases goes directly to heat the atmosphere (Greenhouse feedback), whereas in the tropics most of the extra energy is consumed by evaporation. Thirdly, the thickness of the layer of the atmosphere, which must be heated to provide heating of the near surface air, is much less in the Arctic than in the tropics, that leads to a significant temperature rise in the Arctic. Fourth, since warming leads to sea ice loss, solar heat absorbed by the ocean in the summer season is more easily transferred to the atmosphere in the winter, causing a significant increase in air temperature. Finally, since the heat is transferred to the Arctic both by the atmosphere and oceans, changes in the structure of the

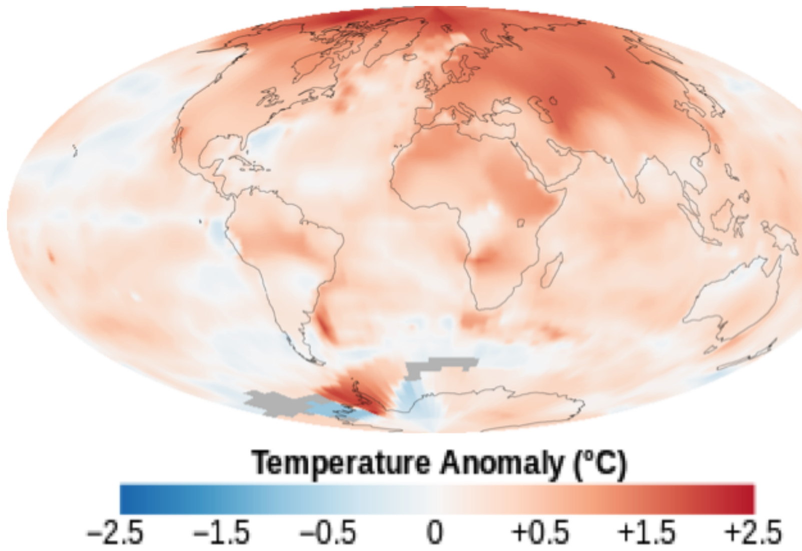


Figure 4.3: The map shows the 10–year average (2000–2009) global mean temperature anomaly relative to the 1951–1980 mean. The largest temperature increases are in the Arctic [Voiland, 2010].

atmospheric and oceanic circulation can also lead to additional warming of the Arctic [Hassol].

4.3.1 Past Arctic tipping points

The rapid warming in the northern part of the Earth is known as Arctic amplification (AA). Many studies and researches were made to examine the reasons of this phenomenon. Graversen [Graversen et al.] found out that warming in the surface was less than on the upper air over the period 1979–2001, which led them to conclude that the Arctic sea ice loss was not the main driver of AA, but atmospheric poleward energy transport. However, Serreze [Serreze et al., 2009] with Screen and Simmonds [Screen and Simmonds, 2010b] analyzed different data sets and made alternative conclusion: strongest AA on the surface was caused by changes in the surface energy balance because of the loss of the Arctic ice. It can be explained by an increase in the rate of sea ice decline. The biggest evidence of the warming Arctic is a widespread melting of glaciers and sea ice, and a reduced period of the biggest snow coverage. According to some model estimates, during this century,

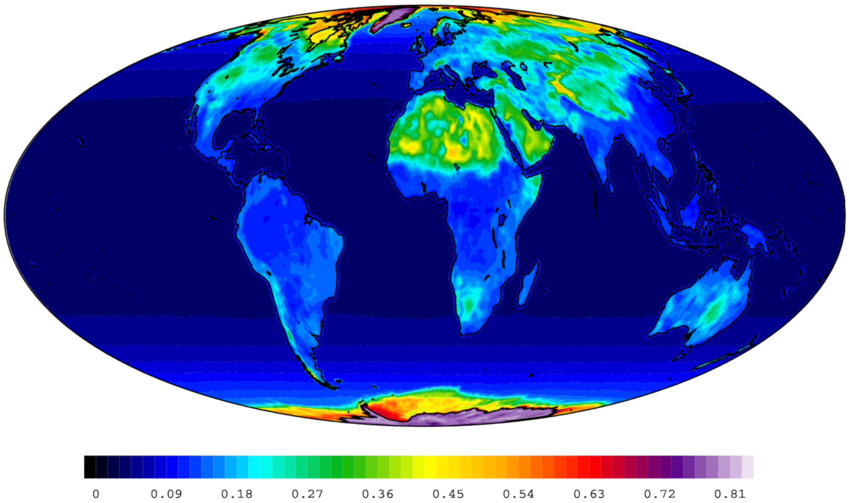


Figure 4.4: Albedo of the Earth's surface measured by the satellite. Data collected from the period April 7–22, 2002, by NASA Earth Observatory [Budikova et al.]. The global albedo is about 40%. The biggest albedo is for snow and ice surfaces — 70–90%. 35% is a sand albedo, grass cover reflect 20–25% of solar radiation, and a forest area albedo is equal to 5–20%. The lowest albedo is for ocean and seas — 5%.

the growth of global carbon dioxide and other greenhouse gases emissions from human activities (primarily, the burning of fossil fuels) will additionally contribute to the warming of the Arctic to 4–7°C [Stocker et al.]. Impacts of such climate changes in the Arctic will dramatically influence all the planet. As a consequence of this fast warming in the Arctic, ice coverage of the Arctic ocean was at its lowest level ever recorded in 2007 [Allison et al.], the Greenland loses ice sheet mass at an impetuous rate [Rignot et al.], permafrost in Alaska is melting rapidly and forming lakes [Jorgenson et al.]. These and some more (Atlantic thermohaline circulations and boreal forest) are named as potential tipping elements in the Earth system, climate subsystems which could pass a tipping point [Lenton, 2012].

The minimum ice coverage of the Arctic Ocean has decreased obviously during the last decade, the greatest ice melt took place in 2007, then the second lowest areal coverage decline was in 2008, the fourth and the third — in 2009 and in 2010 respectively [Lenton, 2012]. Winter sea ice extent has declined 1.5 million km^2 during the decade 1997–2007 [Nghiem et al.]. There is also observable thickness of the ice cap from 3.6 to 1.9 meters over 1987–2007 [Wadhams].

During 1979–2007, 85% of the Arctic region absorbed up to 5% per year more solar radiation causing a vast bottom ice melt [Perovich et al.]. Sea ice melt and increased albedo became the reasons of even more warming of the lower atmosphere in the Arctic [Screen and Simmonds, 2010a].

4.3.2 Potential Arctic tipping points

Some models [Lenton, 2012] claim that the Arctic may have already been involved in an irreversible process when a large part of the ocean becomes ice-free in summer starting with 2016 ± 3. However, according to Holland [Holland et al.], Boe [Boe et al.] and Tietsche [Tietsche et al.] possible tipping points mostly take place around 2050. Some authors [Eisenman and Wettlaufer] argue that a year-round sea-ice loss is more likely to present a tipping point, which requires 13°C warming at the North Pole. This can happen during 21st century only under very high anthropogenic forcing, which is not very possible [Lenton, 2012].

Chapter 5

Early warning signals and tipping points

In the following chapter, we will consider two examples of a tipping point model with white noise and fractional Brownian motion. We will examine how variance and correlation plots behave prior bifurcation.

5.1 Data

The datasets of global temperature used in next chapters are obtained by Climatic Research Unit (CRU) at the University of East Anglia, UK. The data is given for the period January 1850 – January 2015 and for each month there are 11 data records of mean temperature on the planet [cru.uea.ac.uk].

We will use the sea ice extent data records from National Snow and Ice Data Center (NSIDC) at the University of Colorado, Boulder, USA [nsidc.org]. The data is given daily in degrees Celsius for the period from November 1978 to January 2015.

The Arctic temperature data records were obtained by International Arctic Research Center at the University of Alaska Fairbanks, USA. The datasets are given monthly in Celsius for different periods of time for each place (in average from 1870 until 2008) [iarc.uaf.edu]. Additionally, the Vardø temperature dataset is obtained by NASA Goddard Institute for Space Studies given for 1880–2015 [giss.nasa.gov].

5.2 Basic example of tipping point with white noise

Let us consider as an example the following dynamical system:

$$y' = r - y|1 - y| + \sigma\omega, \tag{5.1}$$

where ω is an additive noise. This can be written as follows:

$$y' = U'(y) + \sigma\omega,$$

where

$$U(y) = ry + \operatorname{sgn} y(\frac{y^2}{2} - \frac{y^3}{3}) + \theta\frac{1-y}{3}.$$

A solution to the equation $r - y|1 - y| = 0$ gives fixed points. We get two cases:

- $y < 1 \to r - y(1 - y) = 0$. When $r < 0.25$, there is one stable:

  $$y = \frac{1 - \sqrt{1 - 4r}}{2}$$

  and one unstable:

  $$y = \frac{1 + \sqrt{1 - 4r}}{2}$$

  fixed points. At the point $r = 0.25$, they unite into one half-stable fixed point.

- $y > 1 \to r + y(1 - y) = 0$. There is just one fixed point:

  $$y = \frac{1 + \sqrt{1 + 4r}}{2}.$$

  That is because $y > 1$, then $1 \pm \sqrt{1 + 4r}$ should be greater than 1, but it is possible only with a plus sign, since $\sqrt{1 + 4r}$ is always greater than 1. We know that this fixed point always exists and is stable.

We rewrite the equation in the following way:

$$y' = F(y) + \sigma\omega,$$

where ω is white noise. Let y_0 be a fixed point. Then we take an approximation of the function $F(y)$:

$$F(y) = F(y_0) + F'(y_0)(y - y_0) + \dots,$$

but we know that $F(y_0) \to 0$ because y_0 is the fixed point and $F'(y_0) < 0$ because the fixed point y_0 is stable, then the linearized equation will take the form:

$$y' = -a(y - y_0) + noise.$$

This is the Ornstein-Uhlenbeck equation that can be rewritten in the following form:

$$dY(t) = -a(y - y_0)dt + \sigma dB(t).$$

Fixed points become unstable then we have:

$$a \to 0,\ \mathrm{Var}(Y(t)) = \frac{\sigma^2}{2a} \to +\infty.$$

Since it was just an approximation of our equation, it is not a perfect model, but we can expect that variance of the function will increase.

If we look at the plot of the solution to the equation (5.1) that is shown in Figure 5.1(b), we will see that the biggest jump takes place at the time $t = 550$. We assume that this is a tipping point.

As we can see in Figure 5.1(c), an increase of variance takes place before the tipping point starting at time $t = 300$. In Figure 5.1(d), we can observe a big jump on the correlation plot at the time $t = 310$ before the tipping point.

As a conclusion, some people may say that a tipping point in a dynamical system can be obtained only if increases both in variance and autocorrelation are observed.

For example, Ditlevsen [Ditlevsen and Johnsen] shows the significance of both variance and autocorrelation fluctuations for detection of a tipping point using 2 simulations of the Langevin equation:

$$\dot{x} = -\partial_x U_\mu(x) + \sigma\omega,$$

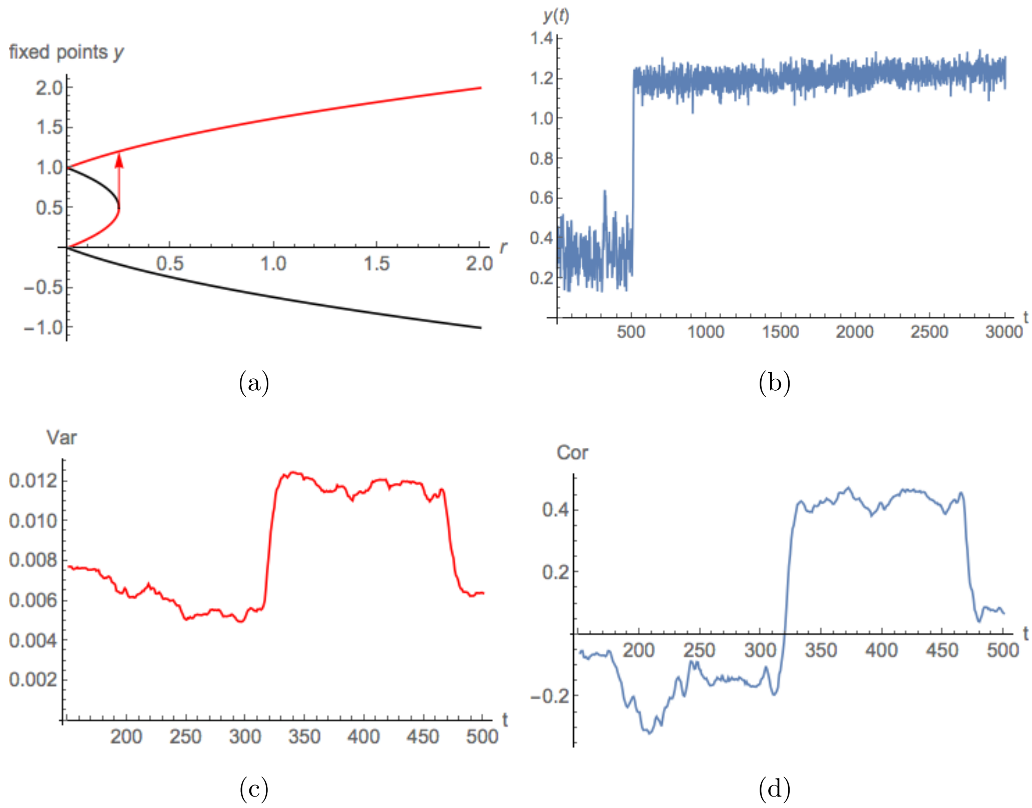


Figure 5.1: Basic model of a bifurcation with a white noise. Bifurcation diagram (a), run of a model with a white noise (b), variance (c) and correlation (d).

where $U_\mu(x) = \frac{x^4}{4} - \frac{x^2}{2} - \mu x$ is a double well potential and ω is white noise. Using two realizations of this model with variable $\mu(t) = \mu_0 * \frac{t}{900}$ and $\sigma = 0.1$ for the first case and constant $\mu = 0$ and $\sigma = 0.25$ for another one, he shows that:

"the early warning of climate changes or structural change in any dynamical system driven through a bifurcation, can only be obtained if increase in both variance and autocorrelation is observed. Conclusion drawn based solely on one of the signals and not the other are invalid" [Ditlevsen and Johnsen].

However, it may not be a good idea to use white noise ($H = 0.5$) in case of working with real data.

5.3 Basic example of tipping point with Brownian motion

We consider a real dataset of sea ice extent for the period 1978–2013. To get rid of the seasonal dependance, we make a plot of annual seasonal cycle of sea ice extent. Then we subtract this cycle repeated for 35 times (Figure 5.2(c)) from dataset of monthly sea ice extent (Figure 5.2(b)), and we get a deseasonalized sea ice loss time series for the period 1978–2013. In order to check if there is a tipping point in this time series, we need to look at the behavior of variance and correlations plots.

In Figure 5.2(e), we can distinguish a jump in variance at the time $t = 410$ that can denote to a true early warning signal. However, some scientists may say that this is just a false alarm due to no visible big jump in correlation plot (Figure 5.2(f)). To check it, we make a model of the sea ice extent time series and apply the analysis we used for the basic model above.

In order to analyze real data records, we need to find the Hurst exponent that denotes to ω. In case with monthly data of sea ice extent, the Hurst exponent is equal to 0.641915 (see Appendix B) that denotes to persistent fractional Brownian motion, not white noise.

We shall in the following investigate variance and autocorrelation of the equation (5.1), where ω is a sample path of Brownian motion with $H = 0.641915$. In this case, there is a tipping point approximately at the time $t = 350$ (Figure 5.3(b)).

Variance plot shows a jump at the time $t = 210$ before the tipping point (Figure 5.3(c)) and in Figure 5.3(d), we see an increase of correlation.

In conclusion, we should mention that such scientists as Lenton [Lenton, 2011] and Scheffer [Scheffer et al.] agree that early warning of an approaching climate tipping point is possible in principle, but there are still problems in detecting true early warning signals. Therefore, we should be careful in our further research so that we do not make wrong conclusions based on missed or false alarms.

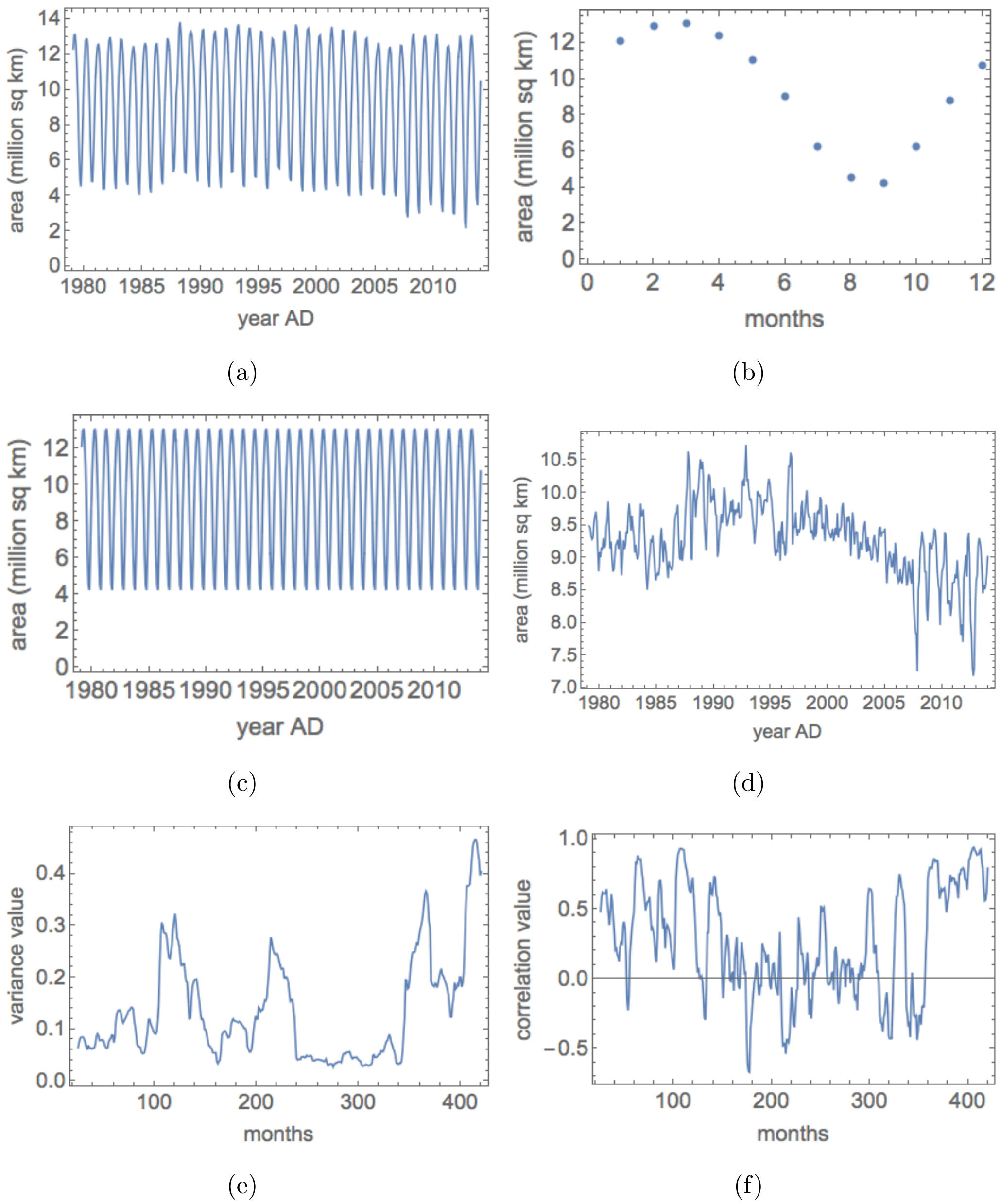


Figure 5.2: Removing seasonal dependance of sea ice extent time series. Plot of real data (a), annual seasonal cycle (b), annual seasonal cycle repeated 35 times (c), deseasonalized sea ice extent time series (d). Variance (e) and correlation (f) of a sea ice extent time series.

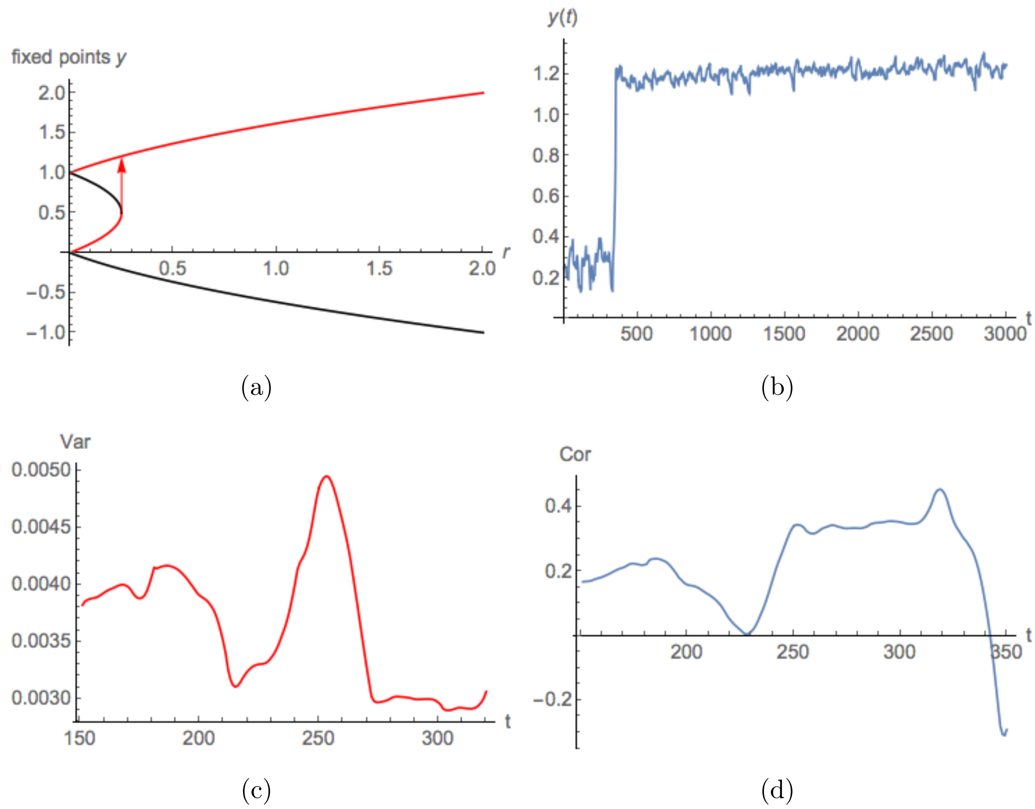


Figure 5.3: Basic model of a bifurcation with fractional Brownian noise. Bifurcation diagram (a), run of a model with a with fractional Brownian noise (b), variance (c) and correlation (d).

5.4 Parameters estimation

In the next two chapters, we will try to find tipping points of sea ice extent in the Arctic using a nonlinear model of future prediction

$$x' = r - x|1 - x|$$

that gives a bifurcation point. In order to explain these observations, we will consider an example of a possible model and show how parameters for these models can be estimated.

An equation of the model is:

$$dx = F_r(x)dt + \sigma dB(t),$$

where

$$F_r(x) = -\frac{1}{a}\left[x_r^2 - x|2x_c - x|\right].$$

Bifurcation occurs for $r = 1$, and before bifurcation we have a stable fixed point:

$$x^* = x_c - x_c\sqrt{1-r}. \tag{5.2}$$

From this, we see that the tipping point occurs when $x \to x_c$. If we analyze data, where the tipping point has occurred, for instance, the Arctic sea ice extent, then x_c can be chosen as the smallest value of the signal prior to the tipping point. Since the fixed point depends on r, which varies with time, then x^* also varies with time. Using the running average:

$$x^*(t) \approx \langle x\rangle_{t,\Delta t} = \frac{1}{\Delta t}\int_{-\Delta t/2}^{\Delta t/2} x(s)ds,$$

and the equation (5.2) we can estimate a control parameter:

$$\hat{r}(t) = 1 - \left(\frac{\langle x\rangle_{t,\Delta t}}{x_c} - 1\right)^2.$$

A linearization around x^* gives:

$$F_r(x) = -\theta(x - x^*) + \mathcal{O}(x - x^*)^2,$$

where

$$\theta = -F_r'(x^*) = \frac{2x_c}{a}\sqrt{1-r}.$$

Then we get a relationship:

$$x^* = x_c + \frac{a}{2}\theta. \tag{5.3}$$

And finally, we get a linear equation:

$$dx(t) = -\theta(x - x^*)dt + \sigma dB(t).$$

This is the Ornstein-Uhlenbeck process with the autocorrelation function $e^{-\theta t}$. Then we have log-dependance relations between the parameter θ and the one-step correlation ρ in the time series:

$$\theta = -\log \rho.$$

We use $\hat{\rho}_{t,\Delta t}$ as an estimate of correlation ρ on windows of length Δt around t and then we get:

$$\hat{\theta}_{t,\Delta t} = -\log \hat{\rho}_{t,\Delta t}.$$

An approximate version of the equation (5.3) is:

$$\langle x \rangle_{t,\Delta t} = x_c + \frac{a}{2}\hat{\theta}_{t,\Delta t}.$$

Then we can use regression for finding an estimate of a. And finally, a parameter σ can be found using the following formula:

$$\sigma = \sqrt{2\bar{\theta}}\,\mathrm{sdev}[x(t) - y(t)],$$

where $\mathrm{sdev}[x(t) - y(t)]$ is the standard deviation of a difference between the real data $x(t)$ and a solution to our equation with no noise $y(t)$ and $\bar{\theta}$ is an average value of all the estimated $\hat{\theta}_{t,\Delta t}$–values.

Chapter 6

Global temperature

In this chapter, we will try to find tipping points of sea ice extent in the Arctic. For that reason, three models with global temperature as a driver of future predictions will be checked: a linear model with white noise and nonlinear models with usage of white noise and Brownian moition as additive noises.

We consider the same dataset of sea ice extent as in the previous chapter, but we will use a different method of removing climatology. The dataset is shown in Figure 6.1(a). The data is given in million square kilometers and it shows how much of sea area is covered by ice for every day during the period from 1 January 1989 until 31 December 2014. Due to winter-summer changes in temperature and, as a consequence, in the amount of ice-covered areas, it is hard to see the real trend of ice extent changes in the sea. In order to get rid of seasonal dependance, we find amplitudes of the data for each year and make a graph of it (Figure 6.1(b)). We make a plot of periodic variations of sea ice area (Figure 6.1(c)). Figures 6.1(b) and 6.1(c) show an increase in the amplitudes of sea ice extent during the given period. Then we subtract periodic variations from the original data set. We divide it in years and find mean value for each year. The following formula:

$$Pure\ Annual\ Sea\ Ice\ Extent =$$
$$\frac{(Real\ Data\ -\ Annual\ Mean\ Value\ of\ Real\ Data) \times Periodic\ Variations}{Mean\ Value\ of\ Periodic\ Variations} +$$
$$Annual\ Mean\ Value\ of\ Real\ Data$$

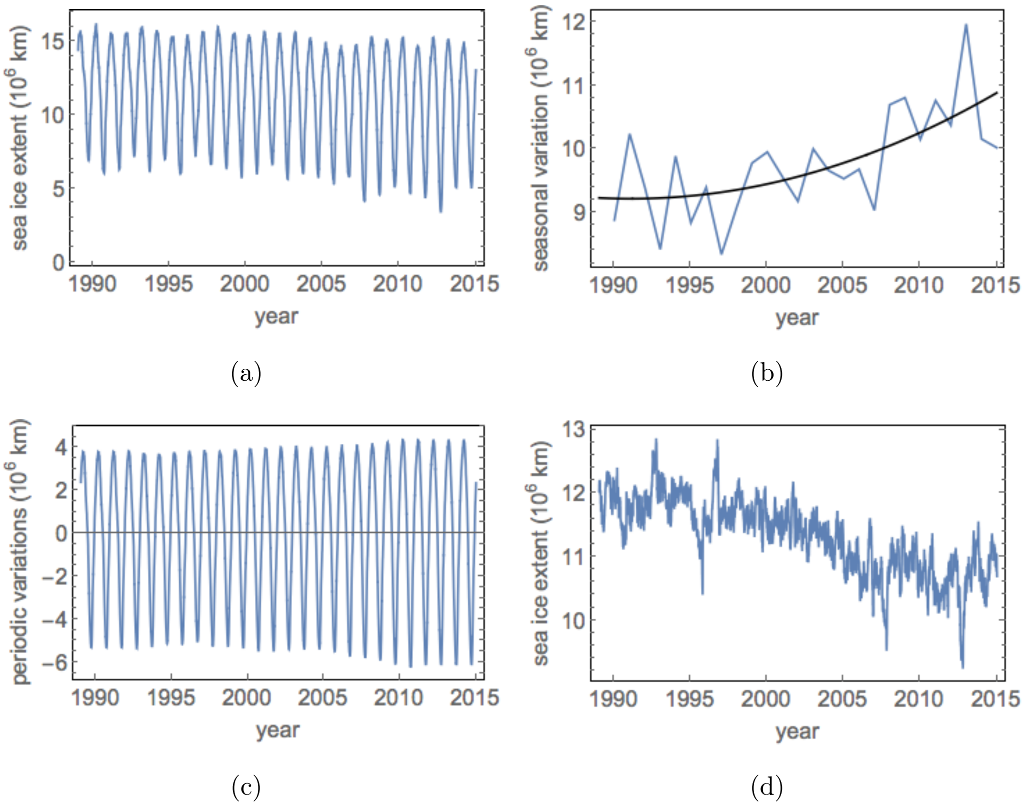


Figure 6.1: Removing seasonal dependance of sea ice extent time series. Real data time series (a), seasonal variation (b), periodic variation (c), deseasonalized sea ice extent (d).

gives pure annual fluctuations of sea ice extent; that is shown in Figure 6.1(d).

6.1 White noise

Let the linear equation $I(t) = A + Bx(t)$ represent a process of sea ice extent changes, where $x(t)$ is a solution to a slightly modified equation (5.1) that takes the following form:

$$\frac{dx}{dt} = a\big(r(t) - x(t)\big|1 - x(t)\big|\big) + \sigma\omega(t), \tag{6.1}$$

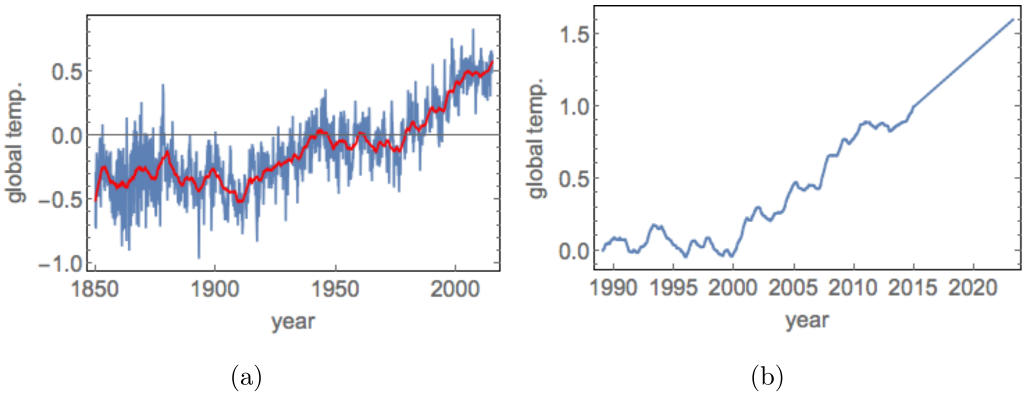


Figure 6.2: Global temperature for each day during the period from January 1850 until January 2015 (a) and global temperature scenario for the next 10 years (b).

where $\omega(t)$ is white noise and $r(t) = r_0 + \nu\langle T_{\text{global}}\rangle_t$ is a driver for sea ice loss, where t_0 corresponds to 1880 and $\langle T_{\text{global}}\rangle_t$ is a dataset of daily global temperature for the period January 1850 – January 2015 (Figure 6.2(a)). A, B, a, σ, r_0 and ν are parameters. A linearized version of the equation (6.1) is:

$$\frac{d}{dx}a\big(r - (1-x)x\big) = a(-1+2x)$$

and since we know that a fixed point for the previous equation is $x_0 = \frac{1}{2}\big(1 - \sqrt{1-4r}\big)$, then we get:

$$\frac{d}{dx}a\big(r - (1-x)x\big) = a(-1+2x)\Big|_{x_0} = a\big(-1 + (1 - \sqrt{1-4r})\big).$$

Applying it to the equation (6.1) gives the linear system:

$$\frac{dx}{dt} = a\sqrt{1-4r}x + \sigma w. \tag{6.2}$$

This is the Ornstein-Uhlenbeck process with $\Theta = a\sqrt{1-4r}$. We find that mean value of r is equal to 0.14, therefore, we can estimate a parameter:

$$a = \frac{\Theta}{\sqrt{1-4r}} = \frac{0,018}{\sqrt{1-4*0.14}} = 0.02752.$$

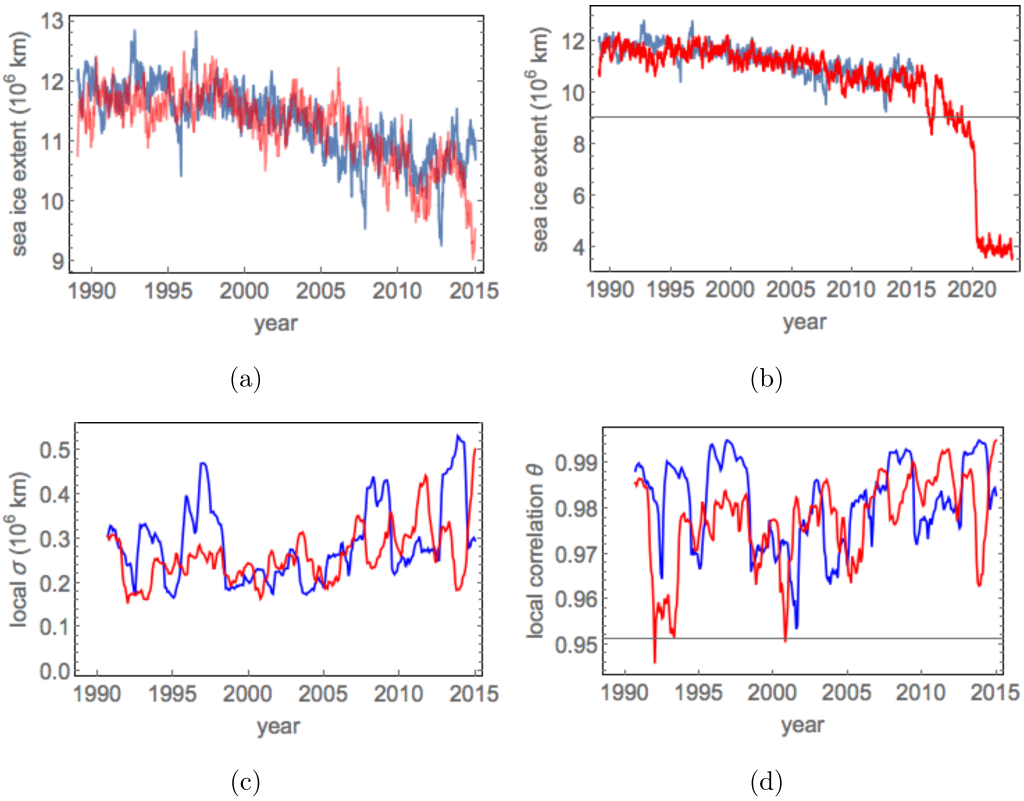


Figure 6.3: Sea ice extent (a) and scenario of sea ice amount for the next 10 years using nonlinear model with white noise as an additive noise (b). Variance (c) and correlation (d) for sea ice extent prediction. Red for model. Blue for real data.

Other parameters can be verified using Mathematica (see Appendix D): $A = 12.5$, $B = 7$, $\sigma = 0.017$, $r_0 = 0.1$ and $\nu = 0.12$.

If we assume that global temperature will keep on rising linearly, for example, as shown in Figure 6.2(b), then we can calculate sea ice extent for the next 10 years. Figure 6.3(b) shows such a prediction and visually we can observe a very rapid drop of sea ice extent from average 8 million km^2 to 4 million km^2 in 2020. We assume that it is a tipping point; but we need to detect early warning signals to prove it. According to the previous chapter, we have claimed that an increase in variance is required for an existence of a tipping point, whilst a rapid jump in correlation is not necessary.

Variance and correlation plots are shown in Figure 6.3(c) and Figure 6.3(d) respectively. Since early warning signals in variance and correlation occur before a tipping point, we need to stop the monitoring not long before a suspected tipping point. The investigated possible tipping point occurs, as we predict, in 2020, then we can stop monitoring of variance and correlation in 2015. We can see the linear trend of an increase in both real data and model variances starting in 2005 That confirms the correct prediction of the tipping point. In addition, there is an observable rise of correlation for the model and real data starting after 2000, it is an extra signal of early warning of the found tipping point.

6.2 Brownian motion

In the previous chapter, the Hurst exponent for sea ice extent time series was found to be equal $H = 0.641915$. We use a sample path of Brownian motion with this Hurst exponent for a model of predicting future possible tipping points of sea ice extent in the Arctic.

As in the previous model, we will use a solution $x(t)$ of the equation (6.1) for the model of sea ice extent:

$$I(t) = A + Bx(t),$$

where A and B are verified parameters. Again we use a scenario of global temperature for the next 10 years (Figure 6.2(b)) as a driver of the model:

$$r(t) = r_0 + \nu\langle T_{\text{global}}\rangle_t,\ t_0 = 1880.$$

Figure 6.4(a) shows the real data time series of sea ice extent during the last 25 years and the model with a sample path of Brownian motion with $H = 0.641915$. We can see that model matches with the real data time series almost perfect. Then we can make a prediction for the next 10 years as shown in Figure 6.4(b) using the given model and the scenario for global temperature.

In the plot of future prediction (Figure 6.4(b)), we can observe a rapid drop of sea ice amount from average 9 million km^2 to 4 million km^2 in 2020. We assume

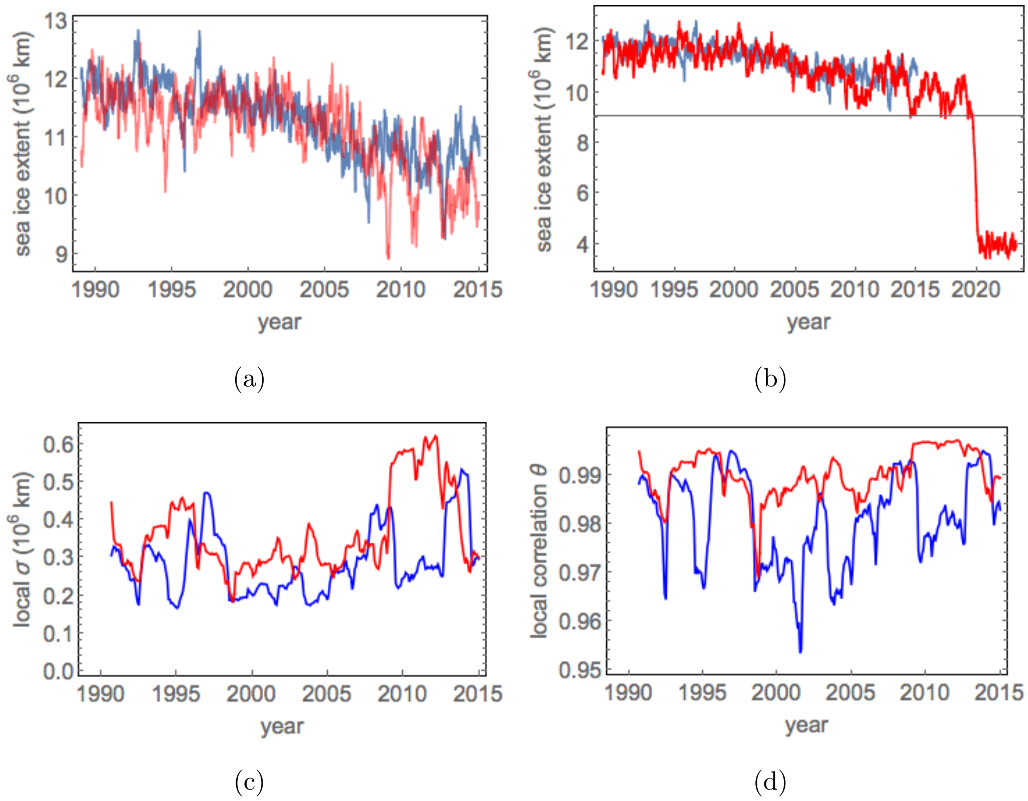


Figure 6.4: Sea ice extent time series (a) and prediction for the next 10 years using a nonlinear model with global temperature as a driver and Brownian noise as an additive noise (b). Variance (c) and correlation (d) for sea ice extent prediction. Red for model. Blue for real data.

that this is a tipping point and try to detect early warning signals.

Plot of variance of the model rises prior to the tipping point, showing early warning signal, especially a big jump at the time $t = 2010$ can be a very good signal of the early warning.

However, in this case, there is a much smaller increase of correlation, but still we can see a linear trend of increasing correlation starting with $t = 1998$.

6.3 Linear model

We also want to check a linear model of a sea ice extent prediction. In this case, the equation (6.1) takes the form:

$$\frac{dx}{dt} = a\big(r(t) - x(t)\big) + \sigma\omega(t), \tag{6.3}$$

with a driver $r(t) = r_0 + \nu\langle T_{\text{global}}\rangle_t$, t_0 corresponds to 1880 and ω is white noise. A solution $x(t)$ to the equation (6.3) will be used in the equation $I(t) = A + Bx(t)$ that models a sea ice extent prediction.

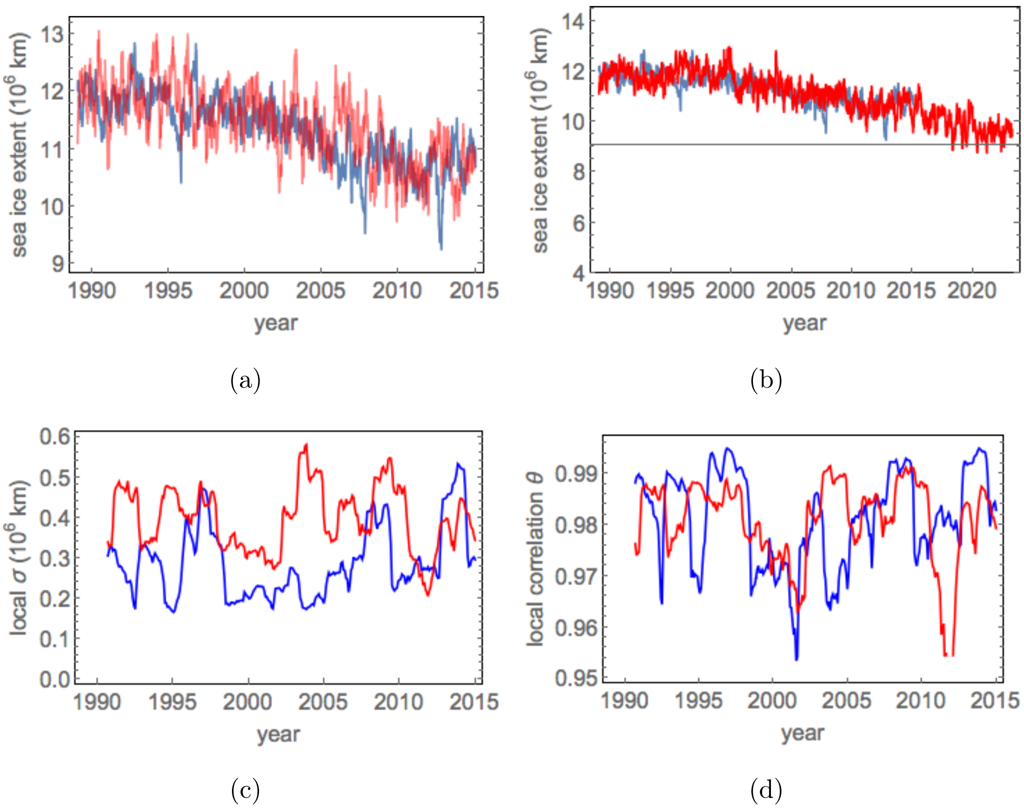


Figure 6.5: Sea ice extent model and real data time series (a), sea ice extent prediction for the next 10 years using the linear model (b). Variance (c) and correlation (d) for sea ice extent prediction. Red for model. Blue for real data.

A fixed point of the equation (6.3) is $x_0 = r$, and then a linear system of the equation (6.3):

$$\frac{dx}{dt} = a\big(r(t) - 1\big)x(t) + \sigma\omega$$

is the OU-process with $a = 0.018$. All of the remaining parameters can be verified using Mathematica (see Appendix H) and they are the following: $A = 21$, $B = 10$, $\sigma = 0.012$, $r_0 = 0.92$ and $\nu = 0.15$.

In Figure 6.5(b), no visible tipping points are observed for the 10 years sea ice extent prediction. But it was expected since this model does not give bifurcation. Increases in variance are false alarms in this case.

Chapter 7

Arctic temperature

Since it was shown that the linear model does not give bifurcation, only the nonlinear model will be considered in this chapter. Yet, we need to find out which noise describes future predictions of sea ice melt better. Then in the following chapter, we consider nonlinear models with white noise and Brownian motion.

As shown in Figure 7.1, temperature in the northern hemisphere has a larger increase during the last century rather than in the southern, and it was said in previous chapters that the Arctic is warming two to four times faster than the rest of the planet, therefore, using global temperature as a forcing of the Arctic sea ice loss cannot give accurate results in search of true early warning and tipping points. It is necessary to use temperature records in the Arctic to investigate early warning signals of sea ice loss in the Arctic.

Observations of mean near surface air temperatures at such places in the Arctic as Alta (Figure 7.2(a)), Tromsø (Figure 7.2(b)), Svalbard (Figure 7.2(c)), Amderma (Figure 7.2(d)), Tiksi (Figure 7.2(e)), and Greenland (Figure 7.2(f)), show a trend of rising temperatures during the last 15–20 years.

For example, we will consider a temperature dataset in Vardø (Northern Norway) as a driver for our models. This dataset was chosen because a good volume of data is available, including data records up to the present time. In Figure 7.3(a), blue lines show a dataset of temperature in Vardø taken for each month during the period from January 1880 until February 2015. Red line shows mean values.

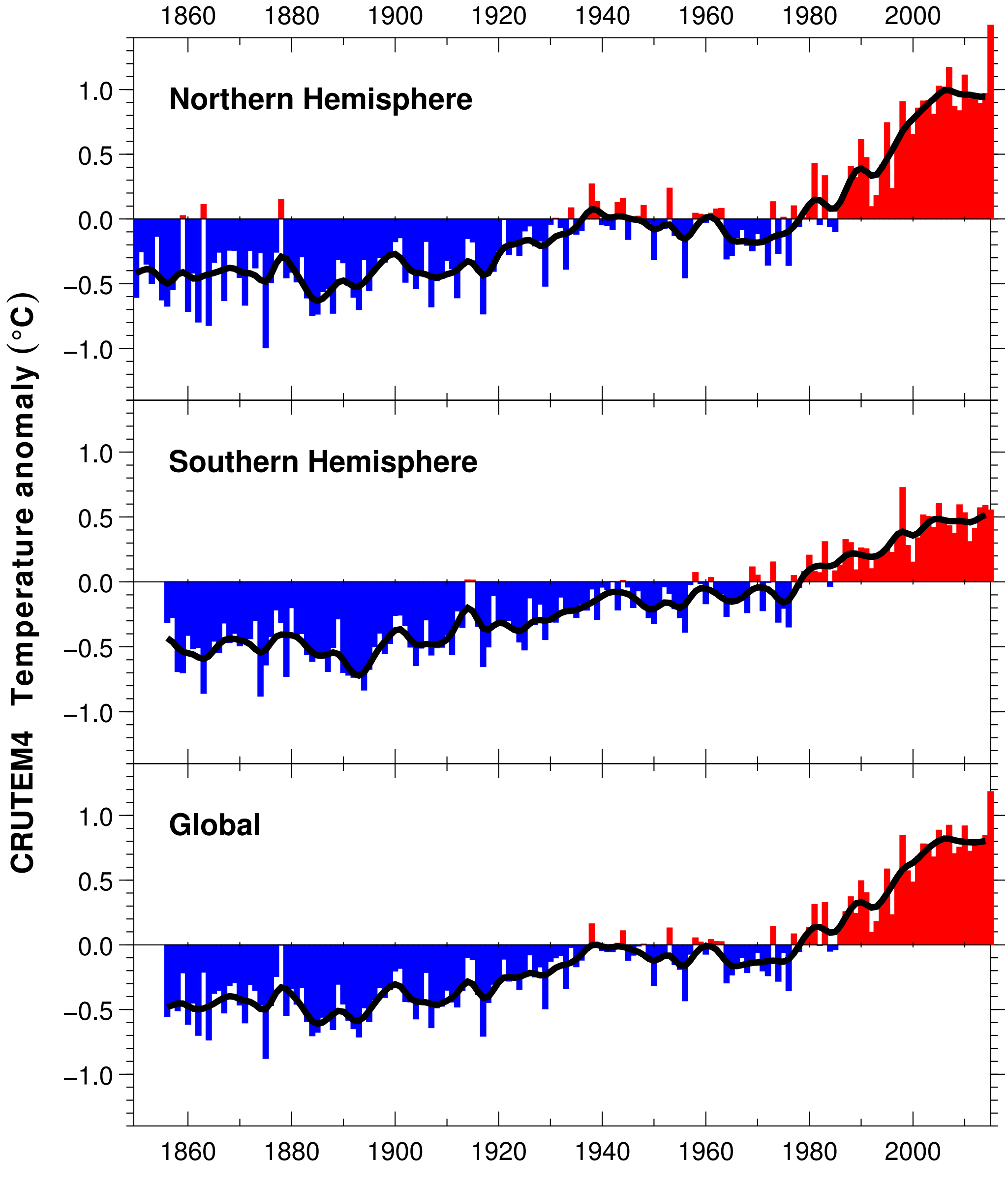


Figure 7.1: Northern and southern hemispheres temperature and global temperature anomalies. Image is courtesy of Climatic Research Unit, University of East Anglia [cru.uea.ac.uk].

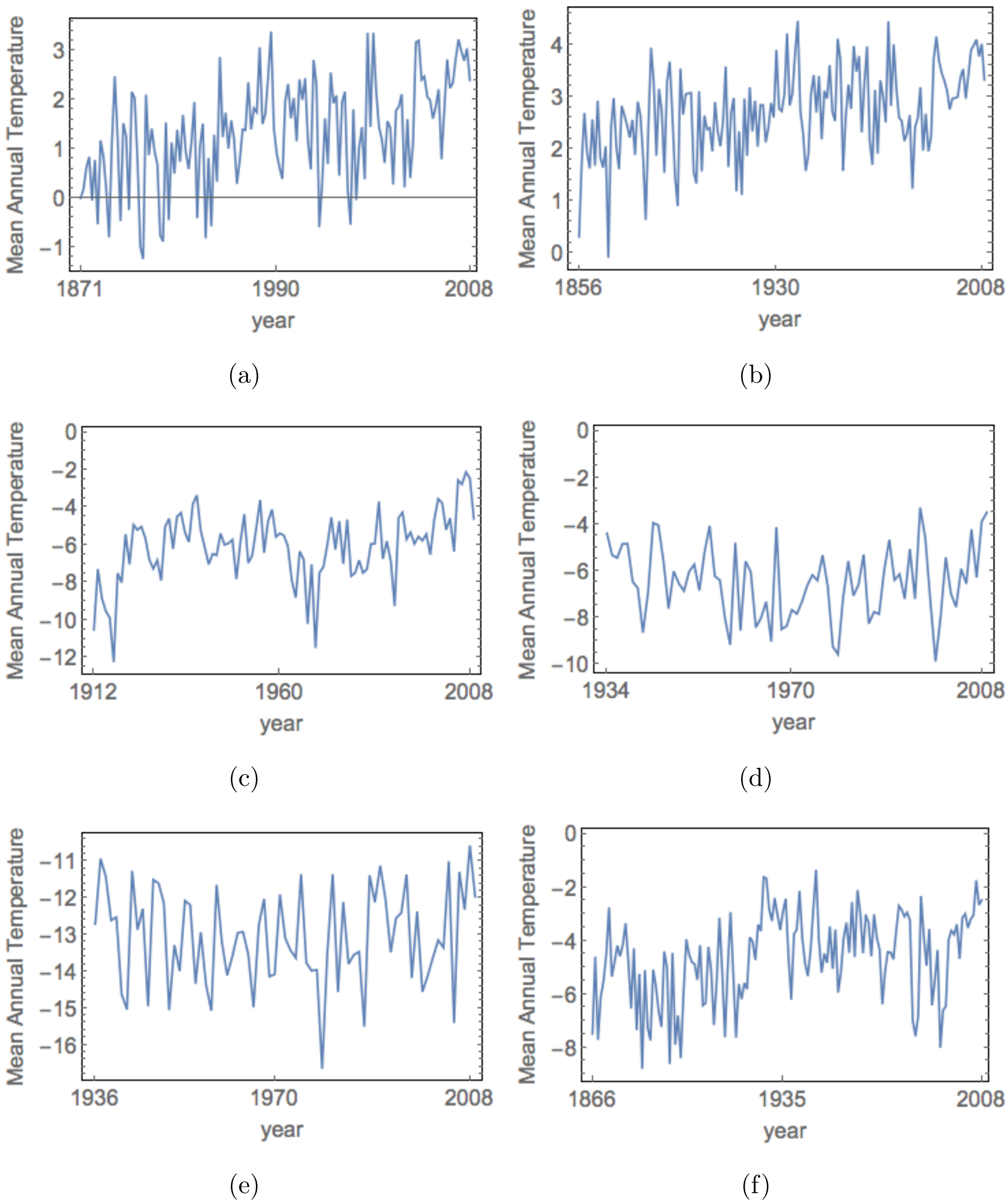
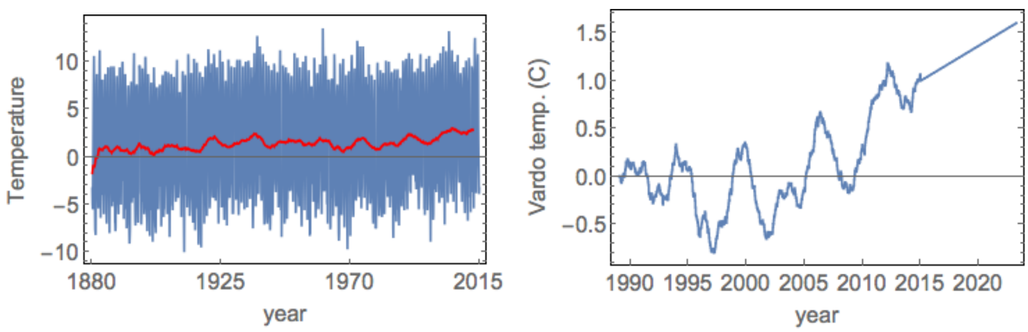


Figure 7.2: Mean annual near surface temperature for Alta (a), Tromsø (b), Svalbard (c), Tiksi (e), Amderma (f) and Greenland (d).

7.1 White noise

For this case, we repeat all the manipulations to get a linearized system (6.2) as in the previous chapter, but we change the driver in the Ornstein-Uhlenbeck process for Vardø temperature instead of global temperature. Parameters verified



(a) (b)

Figure 7.3: Monthly near surface temperature (blue) and moving average (red) (a), and scenario for the next 10 years (b) in Vardø, Norway.

using Mathematica are the following: $A = 12.2$, $B = 7$, $\sigma = 0.017$, $r_0 = 0.1$ and $\nu = 0.11$. Red lines in Figure 7.4(a) are a simulation of the model and blue lines are the real data time series.

If we assume that temperature in Vardø will keep on rising linearly as shown in Figure 7.3(b), then we can calculate sea ice extent up for the next 10 years (Figure 7.4(b)). It is easy to observe a big drop of sea ice extent from average 9 million km^2 to 2 million km^2 taking place in 2022. As seen in Figure (Figure 7.4(b)), the model plot does not match the real data time series.

Variance and correlation are shown in Figure 7.4(c) and Figure 7.4(d) respectively. We again stop the monitoring not long before the suspected tipping point — in 2015.

Two almost simultaneous increases in variance of the model and real data at the time $t = 2005$ and $t = 2013$ denote to an early warning of the tipping point and show that our model matches very well with the real data.

Both model and real data correlation plots show a simultaneous slow rising

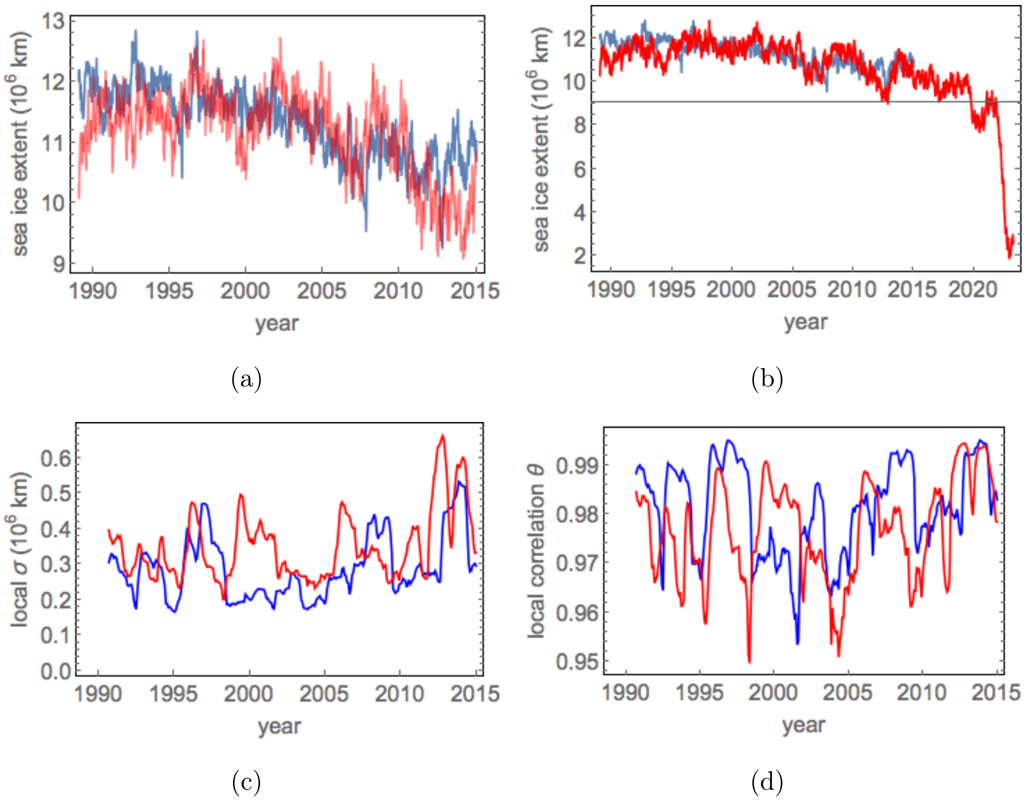


Figure 7.4: Sea ice extent model and real data time series (a), Sea ice extent prediction for the next 10 years using Vardø temperature as a driver and white noise as an additive noise (b). Variance (c) and correlation (d) for sea ice extent prediction. Red for model. Blue for real data.

trend after 2005 that only proves true early warning signals for this model.

7.2 Brownian motion

In case of Brownian motion as an additive noise, model plot shows drop of sea ice extent from average 9 million km^2 to 2 million km^2 over one seasonal cycle in 2021 (Figure 7.5(b)). But again like in the previous section, the model plot does not match the real data plot.

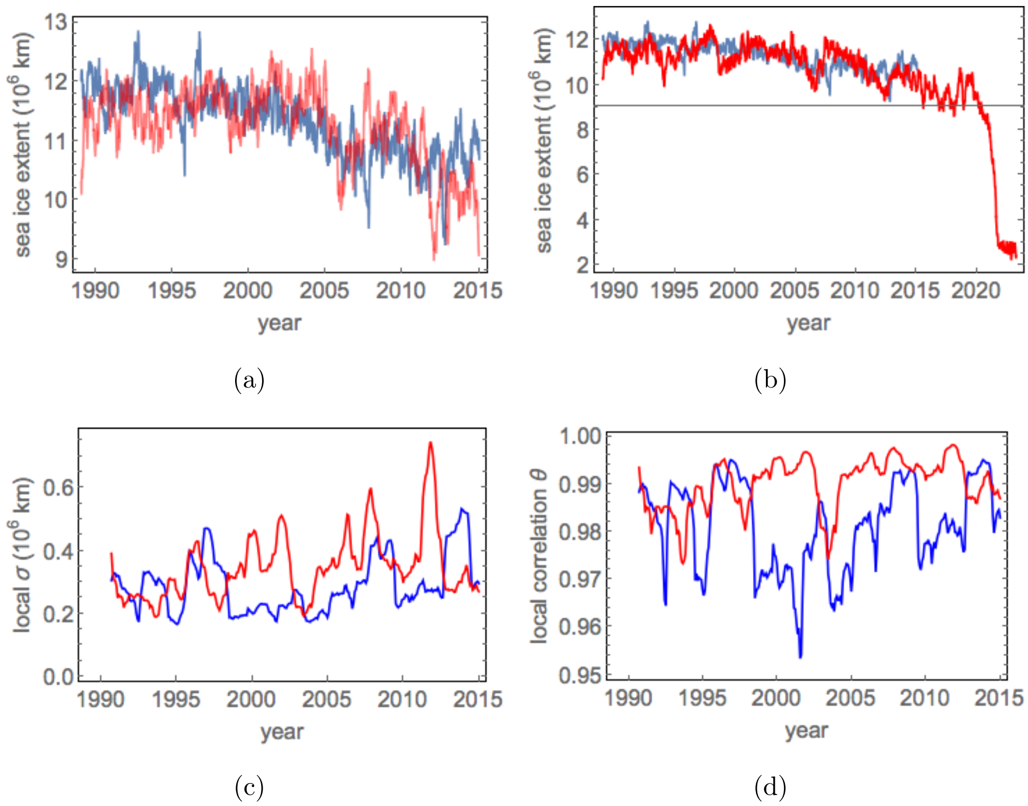


Figure 7.5: Sea ice extent model and real data time series (a), Sea ice extent prediction for the next 10 years using Vardø temperature as a driver and Brownian noise as an additive noise (b). Variance (c) and correlation (d) for sea ice extent prediction. Red for model. Blue for real data.

A jump of variance in 2010 (red line in Figure 7.5(c)) is an early warning signal of the tipping point, however, real data variance does not have such a big rise that can be a sign of a bad model. Moreover, correlation does not show an increase on the interval of 2005–2015 like real data correlation does. These moments can correspond to incorrectness of the model.

Chapter 8

Discussion

8.1 Results

Twenty simulations of sea ice extent and variance plots using Monte-Carlo monitoring system were calculated for each of the models. Using this monitoring, we can estimate sea ice extent and variance amplitudes for the model with 95% assurance.

The first model (Figure 8.1(a)) shows the interval of a possible tipping point for the period 2015-2023. Sea ice will decline from average 11 million km^2 to 4 million km^2 during one seasonal cycle. An increase in variance (Figure 8.2(a)) after 2010 indicates an early warning signal for the tipping point.

The second model (Figure 8.1(b)) gives a prediction for a tipping point on the narrower time interval, approximately, 2015-2021. A loss of ice will make up around 7 million km^2 (from average 11 to 4) over one seasonal cycle. Variance plot (Figure 8.2(b)) gives a big increase before 2015 for the model that proves a true early warning.

The third model (Figure 8.1(c)) starts an amplitude for a possible tipping point in 2020, but does not end before 2025; however, an average line goes through bifurcation in 2023, then we can claim that the amplitude for the possible tipping point is between 2020 and 2027 and sea ice will decline for 9 million km^2 (from 11 to average 2). That is a much heavier loss than in the previous models. Variance plot for the model (Figure 8.2(c)) rises in 2013. This is an early warning signal.

The fourth model (Figure 8.1(d)) shows a decline in sea ice for 9 million km^2 again like in the previous model and the interval, on which the tipping point will take place, from 2019 and an average prediction for the tipping point is for 2022, that gives the end of the interval in 2026. Variance simulations (Figure 8.2(d)) show instability during the entire interval of the monitoring, that gives an assumption that this model does not have true early warning signals of the tipping point on the given interval of time. This can be due to late emergence of the tipping point and the true early warning signal can be detected later in time.

The fifth (linear) model (Figure 8.1(e)) does not show a rapid drop of sea ice extent, but shows a slow decline (due to linearity of the model) from 11 million km^2 to 9 million km^2 during the decade 2015–2025.

8.2 Conclusion

Temperatures in Vardø do not rise even as rapid as those of global, but as we know the Arctic temperature increase is two to four times higher than global. Therefore, the models with usage of Vardø temperatures can not give accurate results, then the models with global temperature are better in predicting future tipping points.

Comparing the models with global temperature shows that model with Brownian motion is better because it corresponds to the real data of sea ice melting in the Arctic and gives more accurate and narrow time intervals for the future possible tipping point.

In a nutshell, a model with global temperature as a driver and Brownian motion as an additive noise predicts the future possible tipping point of sea ice melt in the Arctic best.

8.3 Further work

Predicting sea ice cover changes in the Arctic is a big challenge, in part because it is impossible to make continuous observations of sea ice extent, hence, updated data records can give more accurate results. Also datasets with average Arctic temperature should be used as a driver of sea ice.

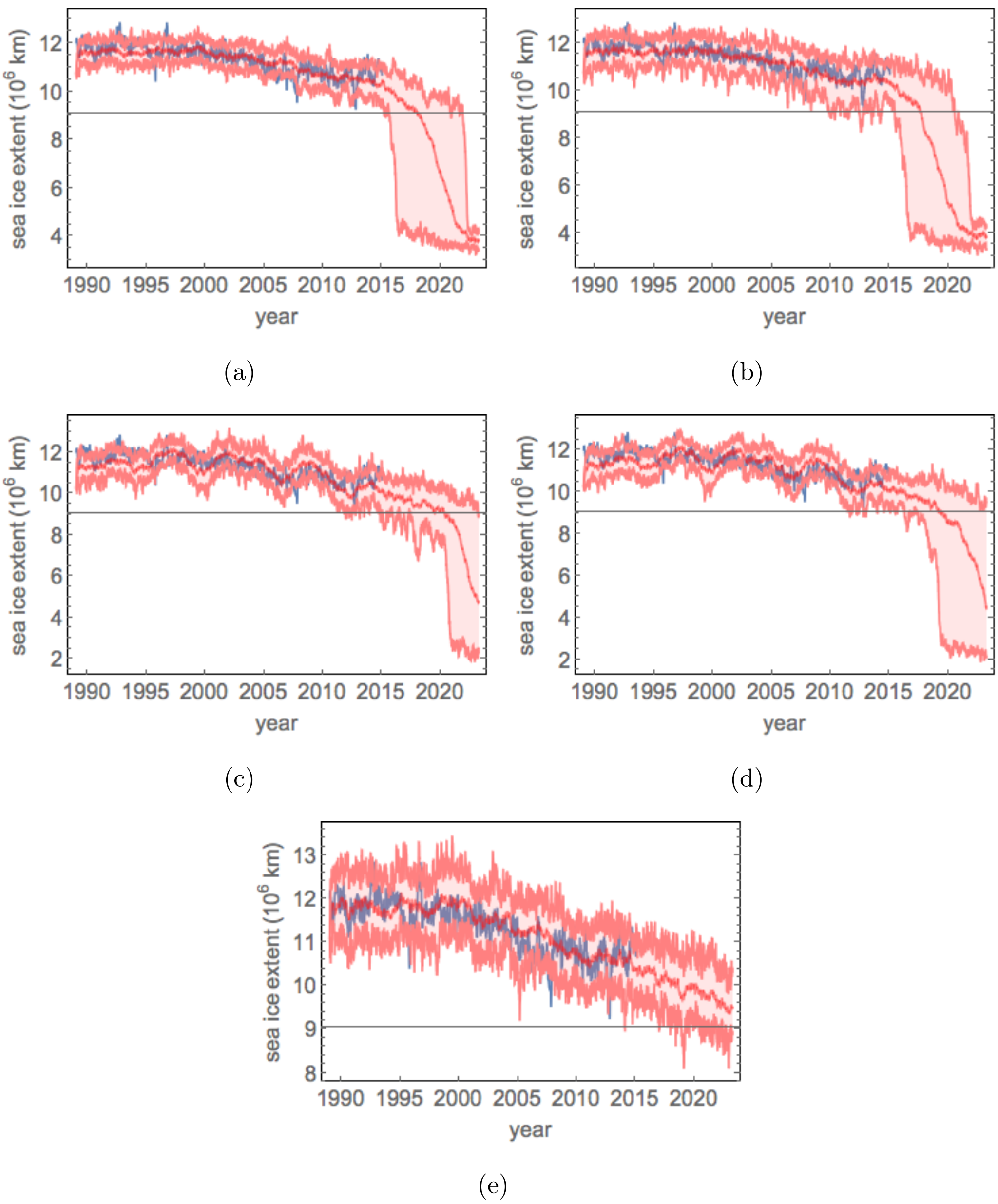
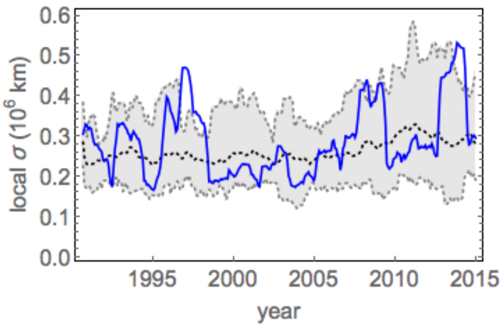
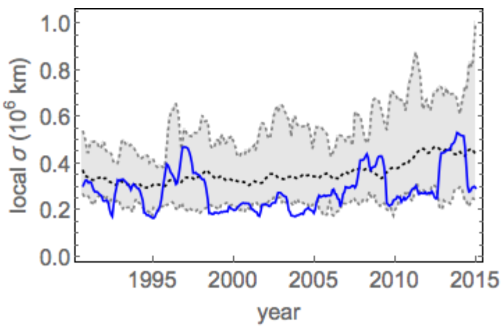


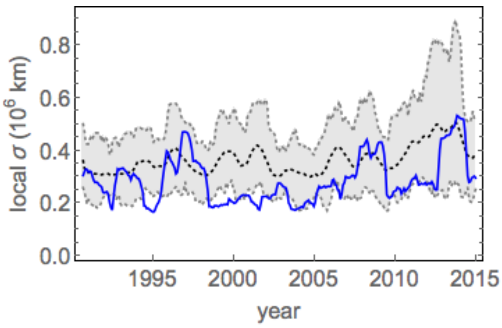
Figure 8.1: Arctic tipping points according to 5 different models: nonlinear models with global temperature and white noise (a), model with global temperature and Brownian noise (b), model with Arctic temperature and white noise (c) and model with Arctic temperature and Brownian noise (d), and the linear model with global temperature and white noise (e). Red for model. Blue for real data.



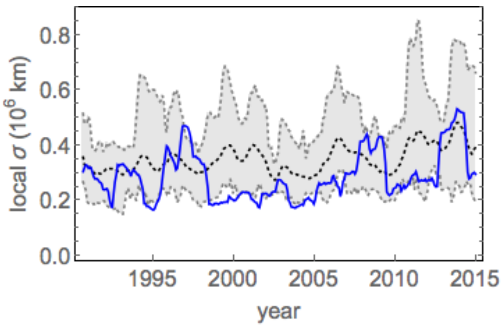
(a)



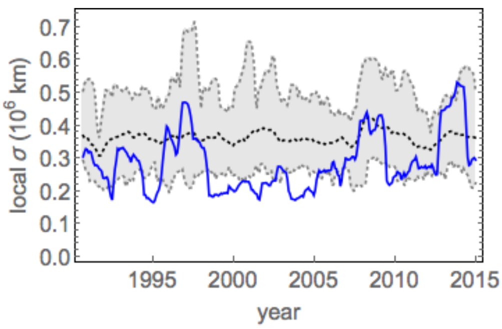
(b)



(c)



(d)



(e)

Figure 8.2: Monte-Carlo simulations of variance for 5 models: nonlinear models with global temperature and white noise (a), with global temperature and Brownian noise (b), with Arctic temperature and white noise (c), with Arctic temperature and Brownian noise (d), and the linear model with global temperature and white noise (e). Grey for model. Blue for real data.

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Appendix A

Example of tipping point with white noise

The following code contains all the calculations and plots for a basic example of a tipping point with white noise as an additive noise discussed section "Basic example of tipping point with white noise" in the chapter "Early warning signals and tipping points". The code was written in the program "Mathematica".

Example of a tipping point

We consider the dynamical system $y' = r - y \mid 1 - y \mid + \sigma w$. This can be written as $y' = U'(y) + \sigma w$, where $U(y) = r\,y + \mathrm{sgn}(y)\left(y^2/2 - y^3/3\right) + \theta(1-y)/3$.

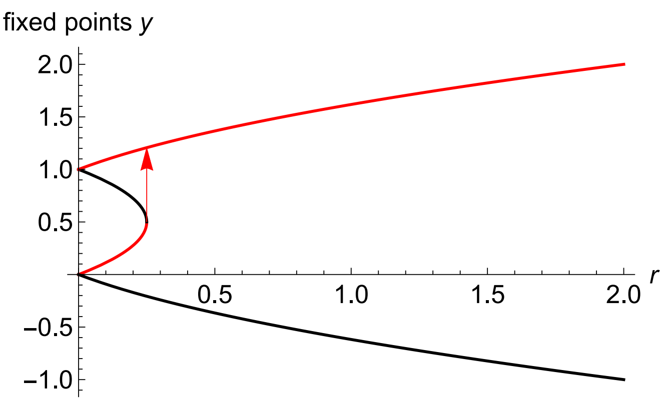
In[263]:=
```
rand = RandomReal[NormalDistribution[0, 0.1], 5 * 31];
rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
ifun = Interpolation[rand];
```

In[266]:=
```
σ = 0.15;
```

In[267]:=
```
i[r_] := r y + Sign[y - 1] (y^2 / 2 - y^3 / 3) + UnitStep[1 - y] / 3;
```
$$f1[r\_] := \frac{1}{2}\left(1 - \sqrt{1 - 4\,r}\right);$$
$$f2[r\_] := \frac{1}{2}\left(1 + \sqrt{1 - 4\,r}\right);$$
$$f3[r\_] := \frac{1}{2}\left(1 - \sqrt{1 + 4\,r}\right);$$
$$f4[r\_] := \frac{1}{2}\left(1 + \sqrt{1 + 4\,r}\right);$$

In[272]:=
```
PL1 = Plot[f1[r], {r, 0, 2}, PlotRange → All, PlotStyle → Red];
PL2 = Plot[f2[r], {r, 0, 2}, PlotRange → All, PlotStyle → Black];
PL3 = Plot[f3[r], {r, 0, 2}, PlotRange → All, PlotStyle → Black];
PL4 = Plot[f4[r], {r, 0, 2}, PlotRange → All, PlotStyle → Red];
line = Graphics[{ Red, Arrow[{{0.25, 0.5}, {0.25, f4[0.25]}}]}];
Show[{PL1, PL2, PL3, PL4, line},
 AxesLabel → {"r", "fixed points y"}, AxesStyle → Directive[14]]
```

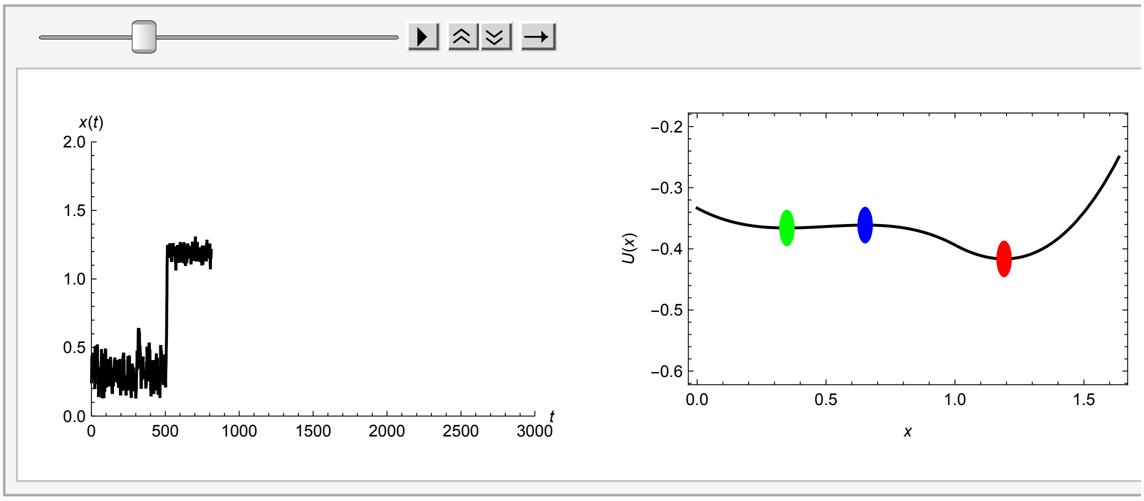
Out[277]=



In[285]:=
```
r = 0.2;
tab = {0.25};
rliste = {r};
pliste = {};
Monitor[
 Do[
   rand = RandomReal[NormalDistribution[0, σ], 5 * 31];
   rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
   ifun = Interpolation[rand];
   (* xxxxxxxx *)
   r = r + 0.001;
   rliste = Append[rliste, r];
   y0 = Last[tab];
   s = NDSolve[
     {y '[x] == r - Sqrt[(1 - y[x]) ^2] * y[x] + ifun[x + 1], y[0] == y0}, y, {x, 0, 30}];
   mid = Flatten[Evaluate[y[x] /. s] /. x → Range[30]];
   tab = Join[tab, mid];
   p1 = ListPlot[tab, PlotRange → {{0, 3000}, {0, 2}},
     Background → None, AxesStyle → Black, PlotStyle → Black,
     Joined → True, AxesLabel → {"t", "x(t)"}];
   PL1 = Plot[-i[r], {y, 0, 1.8}, Background → None, Axes → False,
     Frame → True, FrameStyle → Black, PlotStyle → Black];
   L = {PL1};
   point1 = {f4[r], -i[r] /. y → f4[r]};
   point2 = {f2[r], -i[r] /. y → f2[r]};
   point3 = {f1[r], -i[r] /. y → f1[r]};
   If[Element[point1, Reals],
    L = Append[L, Graphics[{Red, Disk[point1, 0.03]}]];
   ];
   If[Element[point2, Reals],
    L = Append[L, Graphics[{Blue, Disk[point2, 0.03]}]];
   ];
   If[Element[point3, Reals],
    L = Append[L, Graphics[{Green, Disk[point3, 0.03]}]];
   ];
   p2 = Show[L, PlotRange → {-0.6, -0.2}, FrameLabel → {"x", "U(x)"}];

   p = GraphicsGrid[{{p1, p2}}, ImageSize → 600, Background → None];
   pliste = Append[pliste, p];
   , {t, 1, 100}];
 , t]
```

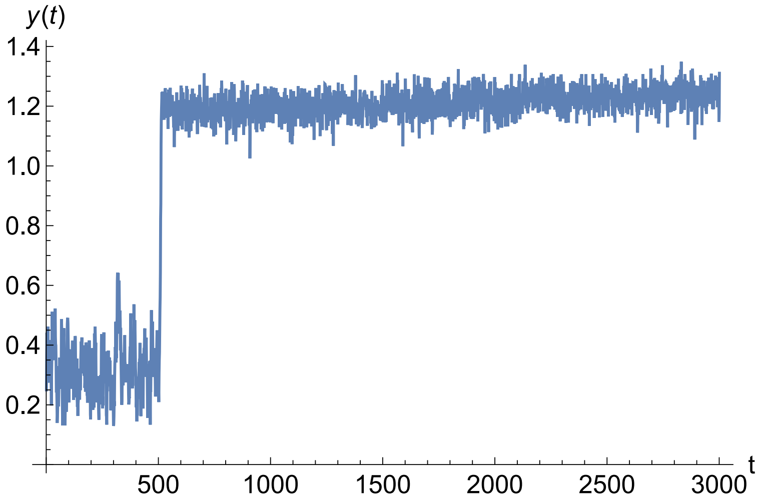
In[290]:=
```
ListAnimate[pliste]
```

Out[290]=



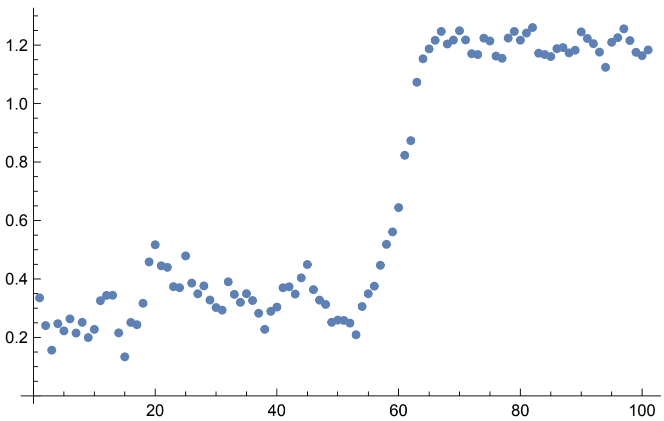
In[292]:=
```
ListPlot[tab, Joined → True, AxesLabel → {"t", "y(t)"},
  AxesStyle → Directive[14], PlotRange → All]
```

Out[292]=



In[299]:=
```
ListPlot[tab[[450 ;; 550]]]
```

Out[299]=



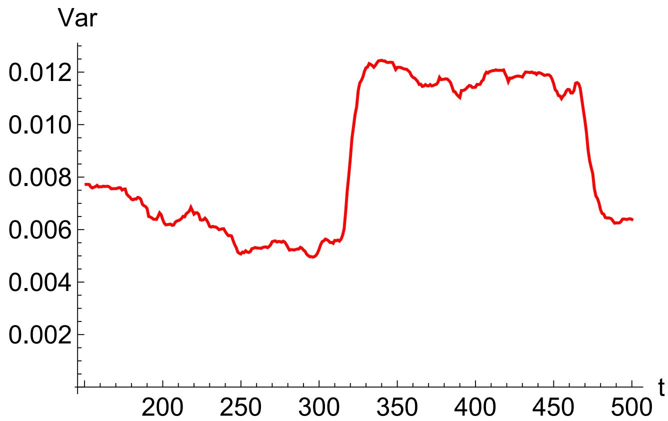
In[300]:=
```
Clear[V]
```

In[301]:=
```
Δt = 150;
V[t_] := Variance[tab[[t - Δt ;; t]]]
```

In[303]:=
```
variancelist = Table[{t, V[t]}, {t, 151, 500}];
```

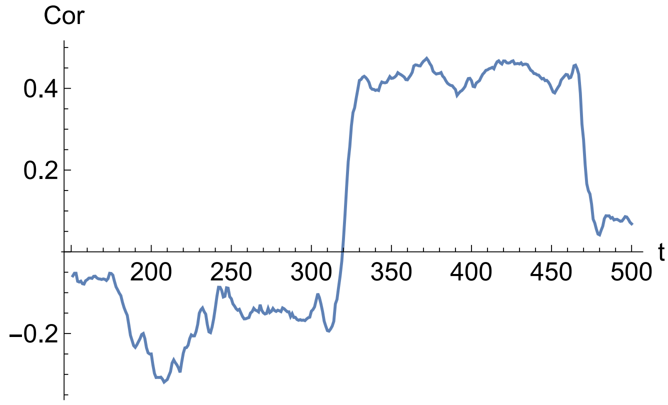
In[304]:=
```
varPlot1 = ListPlot[variancelist, PlotRange → All, PlotStyle → Red,
  Joined → True, AxesLabel → {"t", "Var"}, AxesStyle → Directive[14]]
```

Out[304]=



In[341]:=
```
n = 5;
corlist =
  Table[{t, Correlation[Drop[tab[[t - Δt ;; t]], n], Drop[tab[[t - Δt ;; t]], -n]]},
   {t, 151, 500}];
ListPlot[corlist, Joined → True, PlotRange → All,
 AxesLabel → {"t", "Cor"}, AxesStyle → Directive[14]]
```

Out[343]=

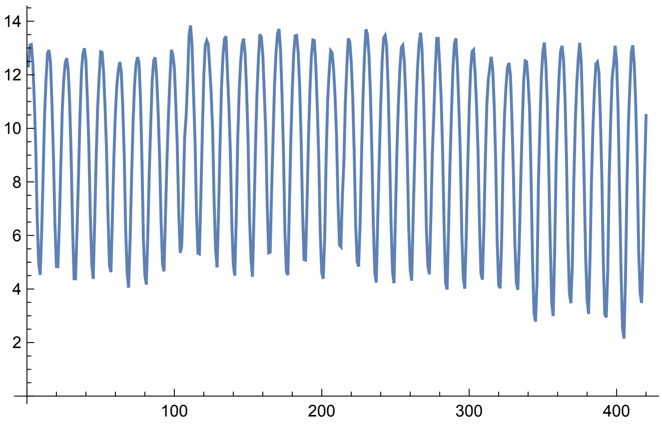


Appendix B

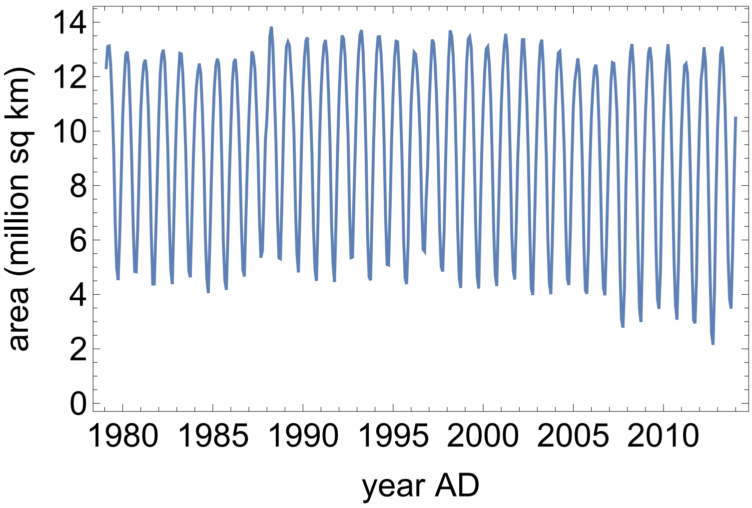
Finding the Hurst exponent

In order to find the Hurst exponent for real sea ice data, we made a set of calculations and plots using Mathematica and in the end of the file we got the Hurst exponent that was discussed in section “Basic example of tipping point with Brownian motion” in the chapter “Early warning signals and tipping points”.

```
seaicemonthly = Flatten[Import["Desktop/Data/SeaIceMonthly.txt", "Table"]];
ListPlot[seaicemonthly, Joined → True, PlotRange → All]
```

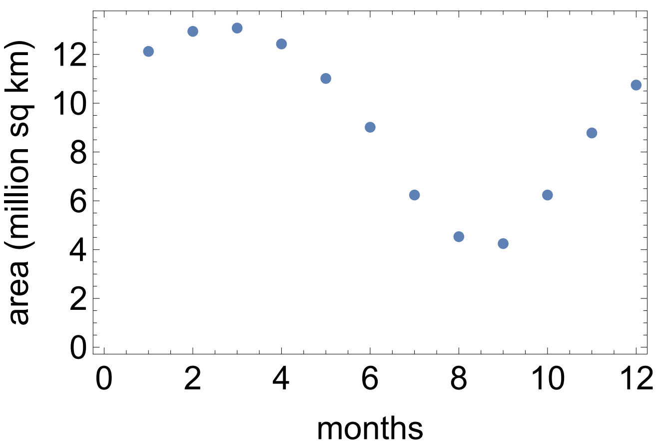



```
seaicemonthly2 =
  Table[{1979 + i / 12, seaicemonthly[[i]]}, {i, Length[seaicemonthly]}];
ListPlot[seaicemonthly2, Joined → True, Frame → True, PlotRange → All,
 FrameLabel -> {"year AD", "area (million sq km)"}, LabelStyle → {18}]
```

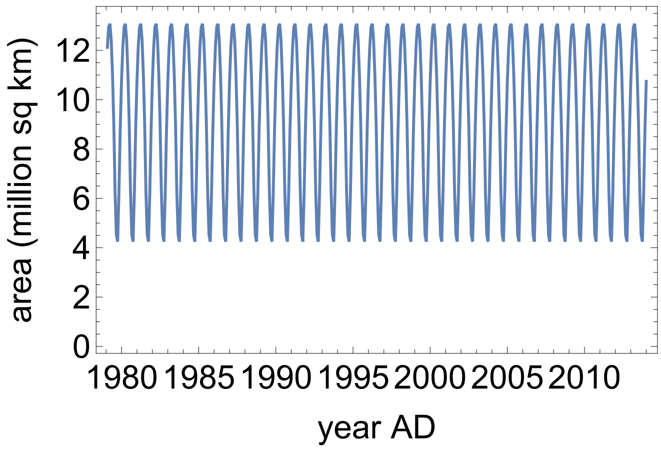



```
year = Map[Mean[#] &, Transpose[Partition[seaicemonthly, 12]]];
```

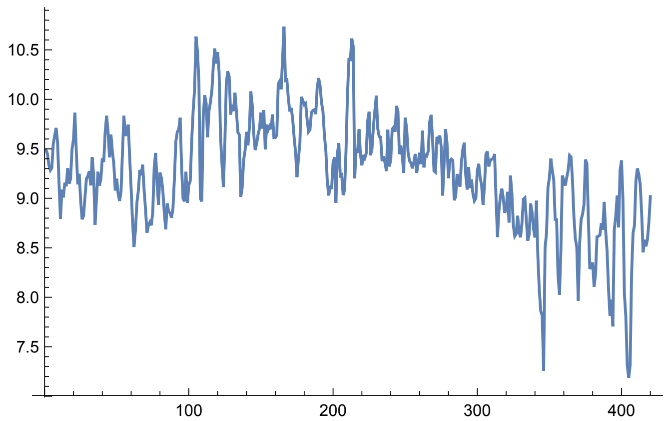
```
ListPlot[year, Frame → True,
 FrameLabel -> {"months", "area (million sq km)"}, LabelStyle → {18}]
```



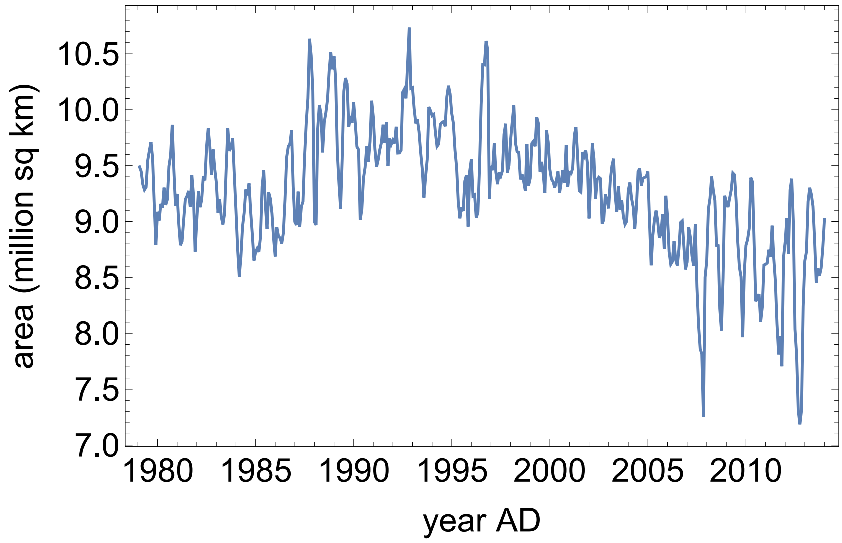
```
v = Flatten[Table[year, {35}]];
v2 = Table[{1979 + i / 12, v[[i]]}, {i, Length[v]}];
ListPlot[v2, Joined -> True, Frame → True,
 FrameLabel -> {"year AD", "area (million sq km)"}, LabelStyle → {18}]
```




```
X = seaicemonthly - v + Mean[v];
ListPlot[X, Joined → True]
```



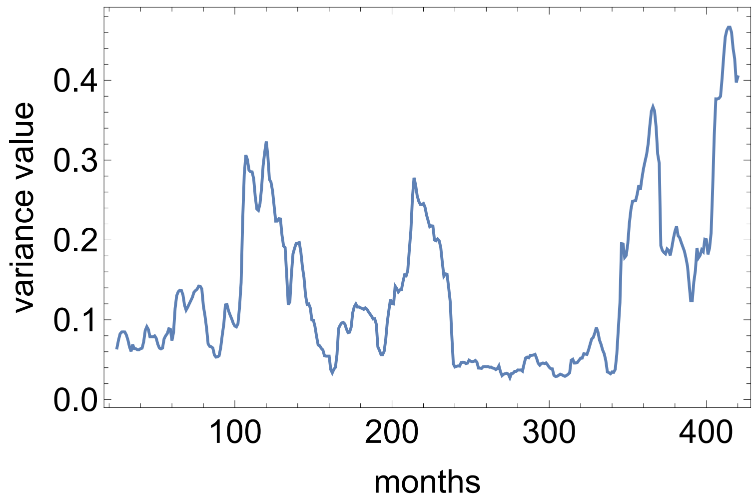
```
X2 = Table[{1979 + i / 12, X[[i]]}, {i, Length[X]}];
ListPlot[X2, Joined → True, Frame → True, PlotRange → All,
 FrameLabel -> {"year AD", "area (million sq km)"}, LabelStyle → {18}]
```




```
t = 50;
Variance[X[[t - 12 ;; t]]]
```

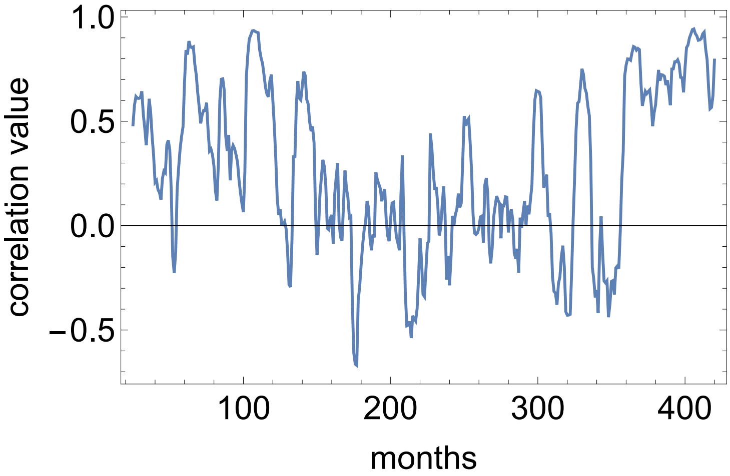
0.0549354

```
variancelist = Table[{t, Variance[X[[t - 24 ;; t]]]}, {t, 25, 420}];
ListPlot[variancelist, Joined -> True, Frame → True, PlotRange → All,
 FrameLabel -> {"months", "variance value"}, LabelStyle → {18}]
```




```
n = 12;
corlist =
  Table[{t, Correlation[Drop[X[[t - 24 ;; t]], n], Drop[X[[t - 24 ;; t]], -n]]},
   {t, 25, 420}];
```

```
ListPlot[corlist, Joined → True, Frame → True, PlotRange → All,
 FrameLabel -> {"months", "correlation value"}, LabelStyle → {18}]
```




```
H = FindProcessParameters[Log[X2], FractionalBrownianMotionProcess[h]]
```

```
{h → 0.641915}
```

Appendix C

Example of tipping point with Brownian motion

The following code contains all the calculations and plots for a basic example of a tipping point with a Brownian motion with the Hurst exponent equal to 0.64 as an additive noise discussed in section “Basic example of tipping point with Brownian motion” in the chapter “Early warning signals and tipping points”. The code was written in the program “Mathematica”.

Example of a tipping point

We consider the dynamical system $y' = r - y \mid 1 - y \mid + \sigma w$. This can be written as $y' = U'(y) + \sigma w$, where $U(y) = r\,y + \text{sgn}(y)\left(y^2/2 - y^3/3\right) + \theta(1-y)/3$.

In[200]:=
```
data = RandomFunction[FractionalBrownianMotionProcess[.64], {0, 5 * 31, 1}];
data = Drop[data["Path"][[All, 2]], 1];
data = Thread[{#1, #2} &[Range[5 * 31] / 5., data]];
ifun = Interpolation[data];
```

In[204]:=
```
Length[Range[5 * 31] / 5.]
```

Out[204]= 155

In[210]:=
```
i[r_] := r y + Sign[y - 1] (y^2 / 2 - y^3 / 3) + UnitStep[1 - y] / 3;
```

$$f1[r\_] := \frac{1}{2}\left(1 - \sqrt{1 - 4\,r}\right);$$

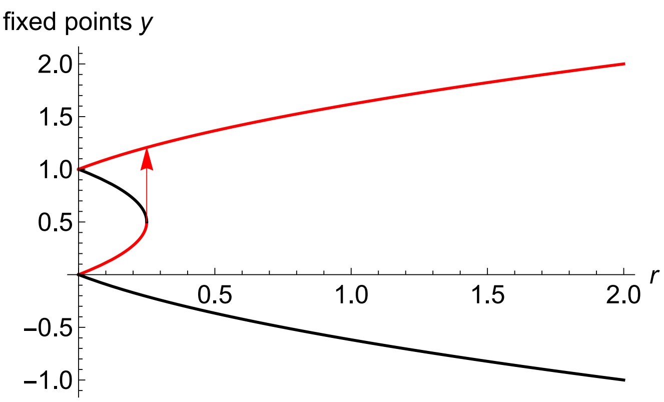
$$f2[r\_] := \frac{1}{2}\left(1 + \sqrt{1 - 4\,r}\right);$$

$$f3[r\_] := \frac{1}{2}\left(1 - \sqrt{1 + 4\,r}\right);$$

$$f4[r\_] := \frac{1}{2}\left(1 + \sqrt{1 + 4\,r}\right);$$

In[215]:=
```
PL1 = Plot[f1[r], {r, 0, 2}, PlotRange → All, PlotStyle → Red];
PL2 = Plot[f2[r], {r, 0, 2}, PlotRange → All, PlotStyle → Black];
PL3 = Plot[f3[r], {r, 0, 2}, PlotRange → All, PlotStyle → Black];
PL4 = Plot[f4[r], {r, 0, 2}, PlotRange → All, PlotStyle → Red];
line = Graphics[{ Red, Arrow[{{0.25, 0.5}, {0.25, f4[0.25]}}]}];
Show[{PL1, PL2, PL3, PL4, line},
 AxesLabel → {"r", "fixed points y"}, AxesStyle → Directive[14]]
```

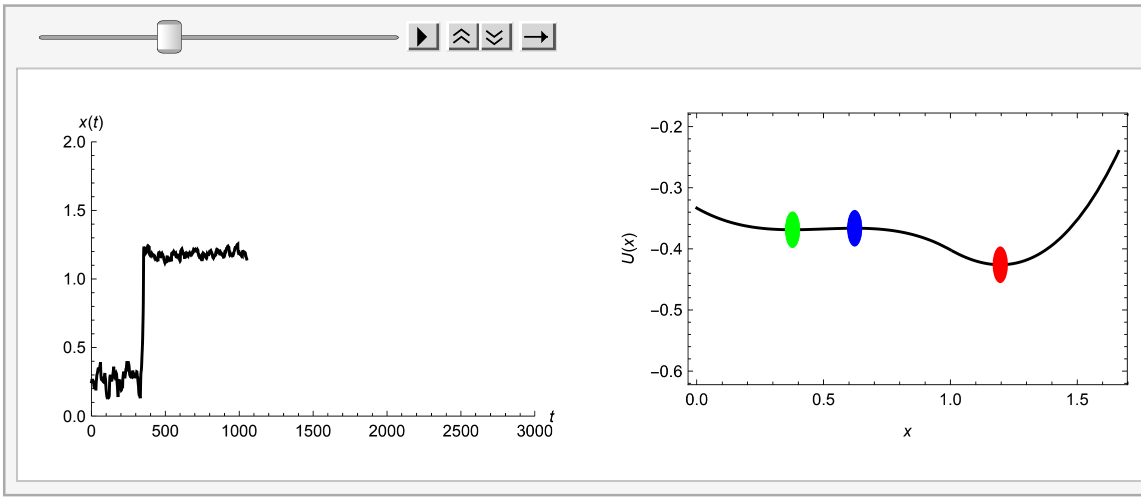
Out[220]=



```
In[221]:= r = 0.2;
tab = {0.25};
σ = 0.002;
rliste = {r};
pliste = {};
Monitor[
 Do[

   data = RandomFunction[FractionalBrownianMotionProcess[.64], {0, 5 * 31, 1}];
   data = Drop[data["Path"][[All, 2]], 1];
   data = Thread[{#1, #2} &[Range[5 * 31] / 5., data]];
   ifun = Interpolation[data];

   (* xxxxxxxx *)
   r = r + 0.001;
   rliste = Append[rliste, r];
   y0 = Last[tab];
   s = NDSolve[{y'[x] == r - Sqrt[(1 - y[x])^2] * y[x] + σ ifun[x + 1], y[0] == y0},
     y, {x, 0, 30}];
   mid = Flatten[Evaluate[y[x] /. s] /. x → Range[30]];
   tab = Join[tab, mid];
   p1 = ListPlot[tab, PlotRange → {{0, 3000}, {0, 2}},
     Background → None, AxesStyle → Black, PlotStyle → Black,
     Joined → True, AxesLabel → {"t", "x(t)"}];
   PL1 = Plot[-i[r], {y, 0, 1.8}, Background → None, Axes → False,
     Frame → True, FrameStyle → Black, PlotStyle → Black];
   L = {PL1};
   point1 = {f4[r], -i[r] /. y → f4[r]};
   point2 = {f2[r], -i[r] /. y → f2[r]};
   point3 = {f1[r], -i[r] /. y → f1[r]};
   If[Element[point1, Reals],
    L = Append[L, Graphics[{Red, Disk[point1, 0.03]}]];
   ];
   If[Element[point2, Reals],
    L = Append[L, Graphics[{Blue, Disk[point2, 0.03]}]];
   ];
   If[Element[point3, Reals],
    L = Append[L, Graphics[{Green, Disk[point3, 0.03]}]];
   ];
   p2 = Show[L, PlotRange → {-0.6, -0.2}, FrameLabel → {"x", "U(x)"}];

   p = GraphicsGrid[{{p1, p2}}, ImageSize → 600, Background → None];
   pliste = Append[pliste, p];
   , {t, 1, 100}];
  , t]
```

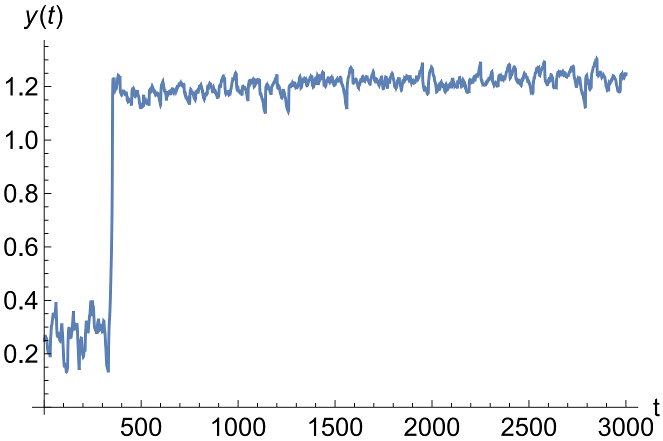
In[227]:=
```
ListAnimate[pliste]
```

Out[227]=



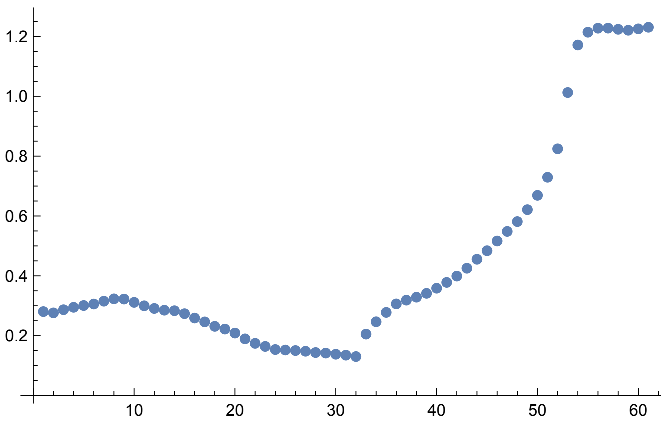
In[228]:=
```
ListPlot[tab, Joined → True, PlotRange → All,
 AxesLabel → {"t", "y(t)"}, AxesStyle → Directive[14]]
```

Out[228]=



In[231]:=
```
ListPlot[tab[[300 ;; 360]]]
```

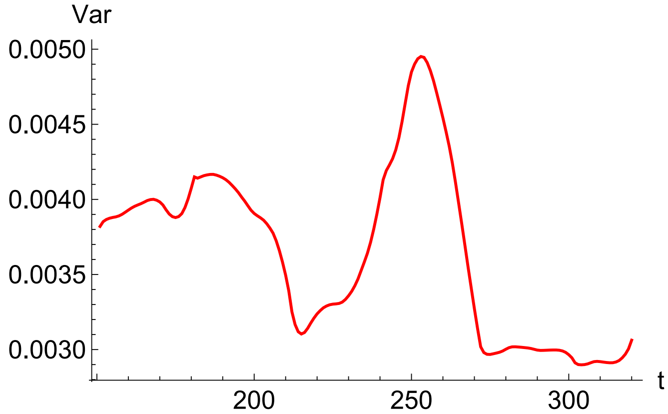
Out[231]=



In[232]:=
```
Clear[V]
```

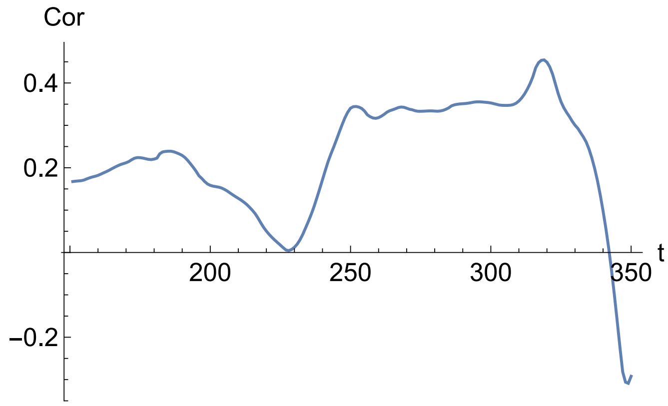
```
In[233]:= Δt = 150;
          V[t_] := Variance[tab[[t - Δt ;; t]]]

In[261]:= variancelist = Table[{t, V[t]}, {t, 151, 320}];

In[262]:= varPlot1 = ListPlot[variancelist, PlotRange → All, PlotStyle → Red,
            Joined → True, AxesLabel → {"t", "Var"}, AxesStyle → Directive[14]]
```

Out[262]=




```
n = 16;
corlist =
  Table[{t, Correlation[Drop[tab[[t - Δt ;; t]], n], Drop[tab[[t - Δt ;; t]], -n]]},
   {t, 151, 350}];
ListPlot[corlist, Joined → True, PlotRange → All,
 AxesLabel → {"t", "Cor"}, AxesStyle → Directive[14]]
```

Out[254]=



Appendix D

Global temperature and white noise

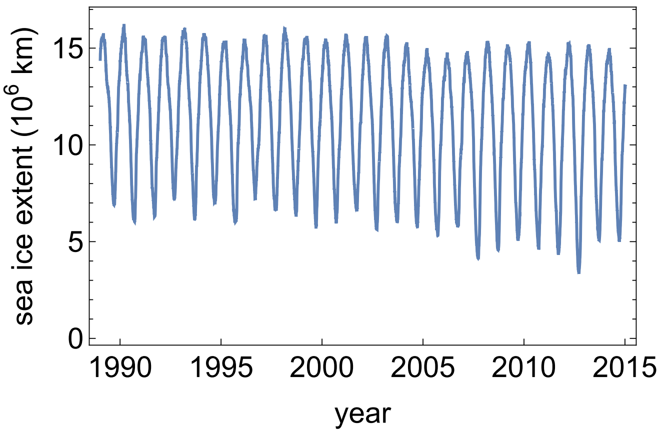
The following code contains all the calculations and plots for real data and modeling of future predictions of sea ice extent in the Arctic using global temperature as a driver and white noise as an additive noise, discussed in section "White noise" in the chapter "Global temperature". The code was written in the program "Mathematica".

Reading and preparing data:

Read sea ice data:

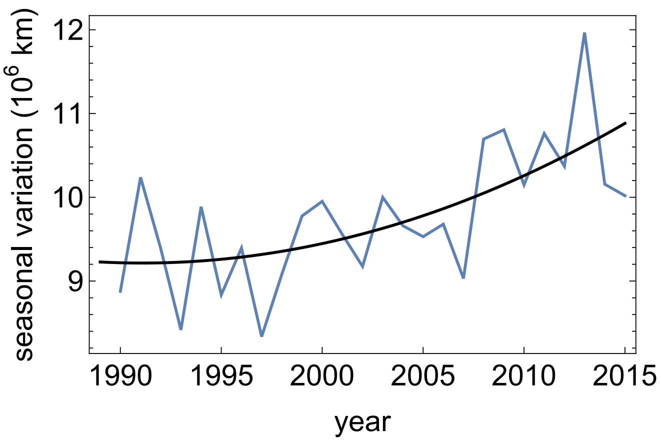
In[1]:=
```
x = ReadList[
    "Dropbox/Master Theses/sea ice/NH_seaice_extent_final.csv", String];
x = Table[ToExpression[StringSplit[x[[t]], ","][[4]]], {t, 3, Length[x]}];
ticks = {{359, "1990"}, {359 + 5 * 365, "1995"}, {359 + 10 * 365, "2000"},
    {359 + 15 * 365, "2005"}, {359 + 20 * 365, "2010"}, {359 + 25 * 365, "2015"}};
y = Drop[x, 2096];
ListPlot[y, Frame → True, FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "sea ice extent (10^6 km)"}]
```

Out[5]=



In[6]:=
```
amplitudes = Map[Max[#] - Min[#] &, Partition[y, 365]];
fit = Fit[amplitudes, {zz^2, zz, 1}, zz];
PL1 = ListPlot[amplitudes, Joined → True];
PL2 = Plot[fit, {zz, 0, 26}, PlotStyle → Black];
newticks = {{1 + 0, "1990"}, {1 + 5, "1995"},
    {1 + 10, "2000"}, {1 + 15, "2005"}, {1 + 20, "2010"}, {1 + 25, "2015"}};
Show[{PL1, PL2}, Frame → True, FrameTicks →
   {{Automatic, Automatic}, {newticks, None}}, FrameStyle → Directive[16],
 FrameLabel → {"year", "seasonal variation (10^6 km)"}]
```

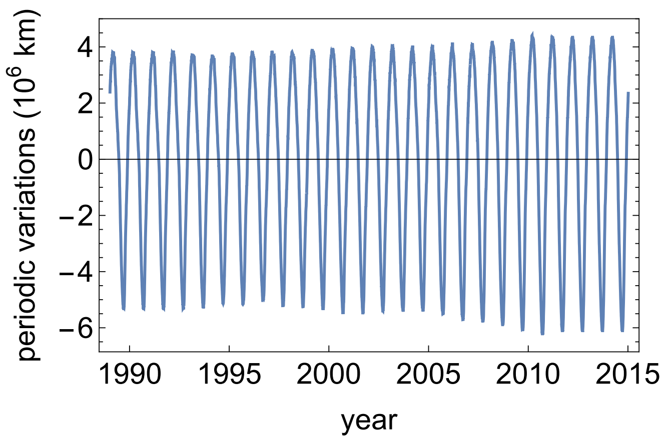
Out[11]=



In[12]:=
```
clim1 = Map[Mean[#] &, Transpose[Partition[y, 365][[1 ;; 5]]]];
clim2 = Map[Mean[#] &, Transpose[Partition[y, 365][[1 ;; 5]]]];
clim3 = Map[Mean[#] &, Transpose[Partition[y, 365][[1 ;; 5]]]];
clim4 = Map[Mean[#] &, Transpose[Partition[y, 365][[2 ;; 6]]]];
clim5 = Map[Mean[#] &, Transpose[Partition[y, 365][[3 ;; 7]]]];
clim6 = Map[Mean[#] &, Transpose[Partition[y, 365][[4 ;; 8]]]];
clim7 = Map[Mean[#] &, Transpose[Partition[y, 365][[5 ;; 9]]]];
clim8 = Map[Mean[#] &, Transpose[Partition[y, 365][[6 ;; 10]]]];
clim9 = Map[Mean[#] &, Transpose[Partition[y, 365][[7 ;; 11]]]];
clim10 = Map[Mean[#] &, Transpose[Partition[y, 365][[8 ;; 12]]]];
clim11 = Map[Mean[#] &, Transpose[Partition[y, 365][[9 ;; 13]]]];
clim12 = Map[Mean[#] &, Transpose[Partition[y, 365][[10 ;; 14]]]];
clim13 = Map[Mean[#] &, Transpose[Partition[y, 365][[11 ;; 15]]]];
clim14 = Map[Mean[#] &, Transpose[Partition[y, 365][[12 ;; 16]]]];
clim15 = Map[Mean[#] &, Transpose[Partition[y, 365][[13 ;; 17]]]];
clim16 = Map[Mean[#] &, Transpose[Partition[y, 365][[14 ;; 18]]]];
clim17 = Map[Mean[#] &, Transpose[Partition[y, 365][[15 ;; 19]]]];
clim18 = Map[Mean[#] &, Transpose[Partition[y, 365][[16 ;; 20]]]];
clim19 = Map[Mean[#] &, Transpose[Partition[y, 365][[17 ;; 21]]]];
clim20 = Map[Mean[#] &, Transpose[Partition[y, 365][[18 ;; 22]]]];
clim21 = Map[Mean[#] &, Transpose[Partition[y, 365][[19 ;; 23]]]];
clim22 = Map[Mean[#] &, Transpose[Partition[y, 365][[20 ;; 24]]]];
clim23 = Map[Mean[#] &, Transpose[Partition[y, 365][[21 ;; 25]]]];
clim24 = Map[Mean[#] &, Transpose[Partition[y, 365][[22 ;; 26]]]];
clim25 = Map[Mean[#] &, Transpose[Partition[y, 365][[22 ;; 26]]]];
clim26 = Map[Mean[#] &, Transpose[Partition[y, 365][[22 ;; 26]]]];
climtab = {clim1, clim2, clim3, clim4, clim5, clim6, clim7, clim8, clim9,
   clim10, clim11, clim12, clim13, clim14, clim15, clim16, clim17, clim18,
   clim19, clim20, clim21, clim22, clim23, clim24, clim25, clim26};
climtab = Map[# - Mean[#] &, climtab];
factors = Map[Max[#] - Min[#] &, climtab];
climseries = Flatten[climtab];
```

In[42]:=
```
ListPlot[climseries, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "periodic variations (10^6 km)"}]
```

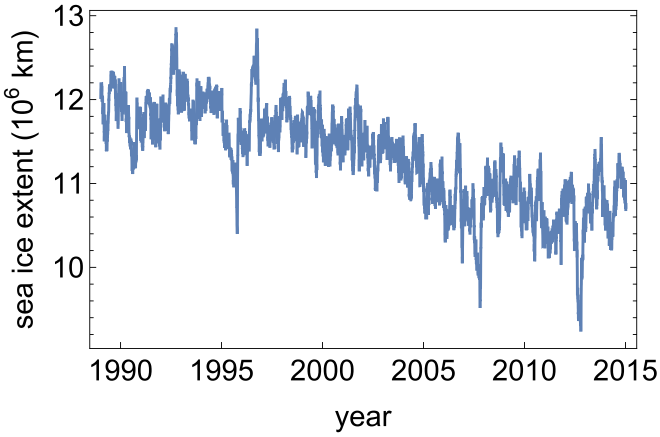
Out[42]=



Remove climatology:

In[43]:=
```
xx = y - climseries;
years = Partition[xx, 365];
yearmeans = Map[Mean[#] &, years];
years = Table[(years[[i]] - yearmeans[[i]]) * (factors[[i]] / Mean[factors]) +
    yearmeans[[i]], {i, 1, 26}];
xx = Flatten[years];
Q1 = ListPlot[xx, Frame → True,
  FrameTicks → {{Automatic, Automatic}, {ticks, None}},
  FrameStyle → Directive[16], Joined → True,
  FrameLabel → {"year", "sea ice extent (10^6 km)"}]
```

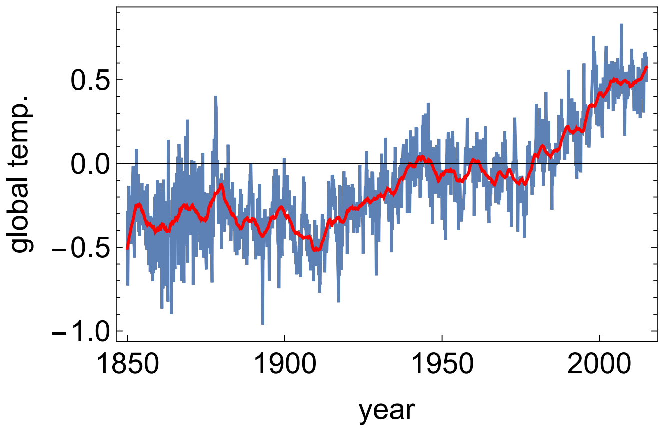
Out[48]=



Global temperature anomaly:

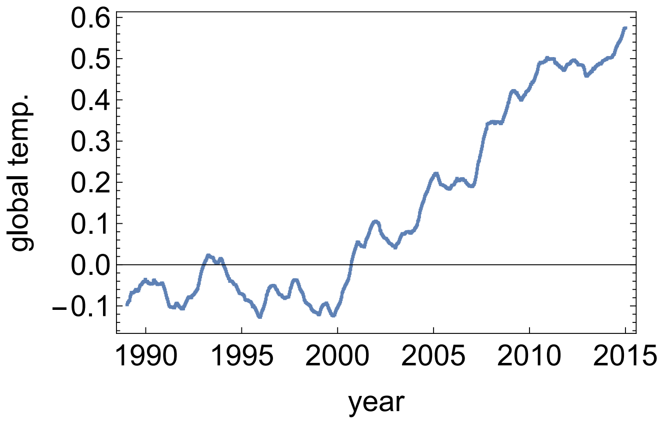
In[49]:=
```
T = ToExpression[
   StringSplit[Drop[ReadList["Dropbox/Master Theses/sea ice/globaltemp.txt",
       String], -1]][[All, 2]]];
ticks2 = {{1, "1850"}, {1 + 12 * 50, "1900"}, {1 + 12 * 100, "1950"},
   {1 + 12 * 150, "2000"}};
PL1 = ListPlot[T, Frame → True, FrameTicks →
     {{Automatic, Automatic}, {ticks2, None}}, FrameStyle → Directive[16],
   Joined → True, FrameLabel → {"year", "global temp."}];
av = Drop[MovingAverage[ArrayPad[T, 5 * 12 / 2, "Fixed"], 5 * 12], -1];
PL2 = ListPlot[av, PlotStyle → Red, Joined → True];
Show[PL1, PL2]
```

Out[54]=



```
In[55]:= R = Flatten[Table[Table[av[[t]], {12}], {t, 1, Length[av]}]];
R = R[[Length[R] - Length[xx] + 1 ;; Length[R]]];
ListPlot[R, Frame → True, FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "global temp."}]
```

Out[57]=



Modeling

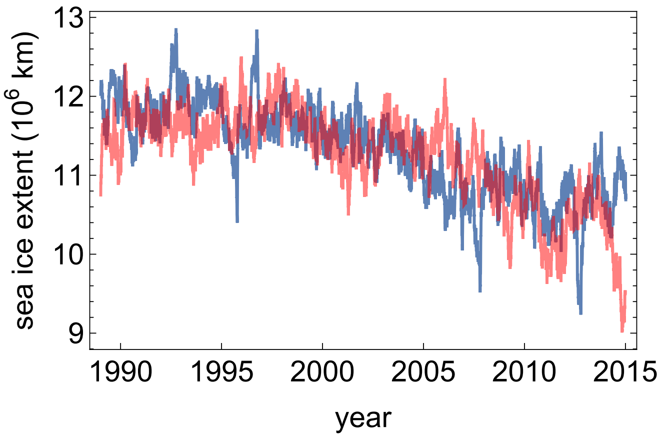
```
In[61]:= OU = xx[[3000 ;; 6000]];
ΔOU = Drop[OU, 1] - Drop[OU, -1];
ΔOU = ΔOU - Mean[ΔOU];
OU = FoldList[Plus, 0, ΔOU];
OU = OU - Mean[OU];
OU = Thread[{#1, #2} &[Range[Length[OU]], OU]];
est = EstimatedProcess[OU, OrnsteinUhlenbeckProcess[μ, Σ, θ]]
```

```
Out[67]= OrnsteinUhlenbeckProcess[0.0000320507, 0.26038, 0.0182579]
```

```
In[68]:= (*0.14 is the mean r-value*)
Clear[a]
sol = Solve[a Sqrt[1 - 4 * 0.14] == est[[3]], a]
```

```
Out[69]= {{a → 0.0275248}}
```

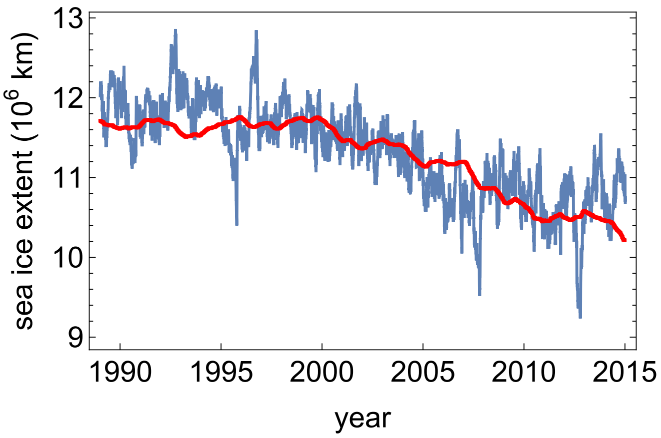
In[70]:=
```
a = 0.02752;
Clear[t];
σ = 0.017;
r0 = 0.1;
v = 0.12;
A = 12.5;
B = 7;
r = r0;
tab = {0.25};
rliste = {r};
pliste = {};
driver = (R - First[R]) / (Last[R] - First[R]);
Monitor[
 Do[
   rand = RandomReal[NormalDistribution[0, σ], 5 * 31];
   rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
   ifun = Interpolation[rand];
   (* xxxxxxxx *)
   r = r0 + v * driver[[30 * t]];
   rliste = Append[rliste, r];
   Q0 = Last[tab];
   s = NDSolve[{Q'[tt] == a * (r - Sqrt[(1 - Q[tt])^2] * Q[tt]) + ifun[tt + 1],
      Q[0] == Q0}, Q, {tt, 0, 30}];
   mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
   tab = Join[tab, mid];
   , {t, 1, 316}];
 , t]
icemodel = -tab;

Q2 = ListPlot[A + B * (icemodel), Joined → True,
   PlotRange → All, PlotStyle → {Red, Opacity[0.5]}];
modelexample = A + B * (icemodel);
Show[{Q1, Q2}, PlotRange → All]
```



How did I find parameters?

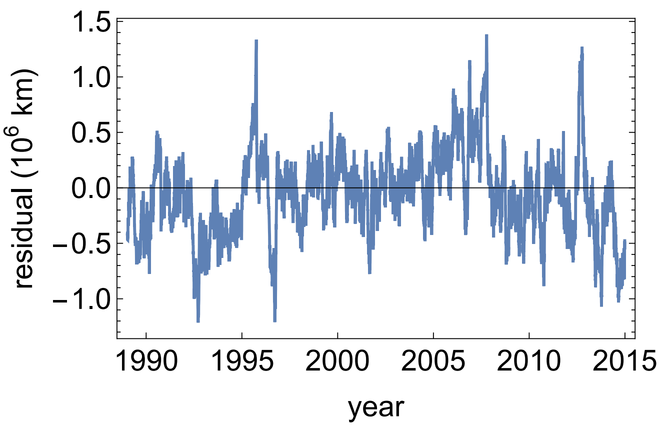
In[84]:=
```
Clear[r]
r0 = 0.1;
v = 0.12;
A = 12.5;
B = 7;
fix[r_] := 1/2 (1 - Sqrt[1 - 4 r]);
Q3 = ListPlot[A - B * Map[fix[#] &, r0 + v * driver[[Range[30 * 316]]]],
    PlotStyle → {Red, Thick}, Joined → True];
Show[{Q1, Q3}, PlotRange -> All]
```

Out[91]=



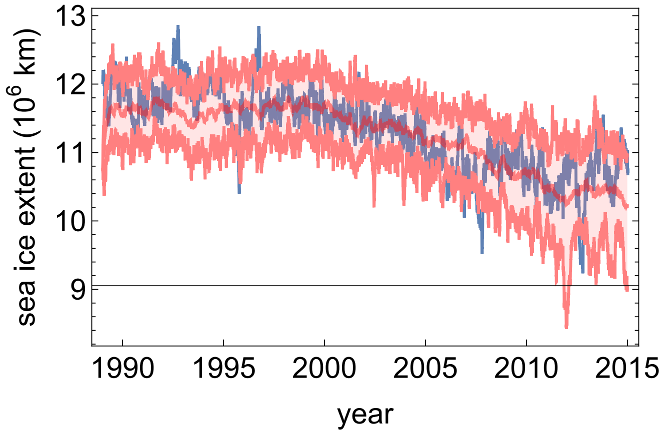
In[92]:=
```
residual = (A - B * Map[fix[#] &, r0 + v * driver[[Range[30 * 316]]]]) - Drop[xx, 10];
ListPlot[residual, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "residual (10^6 km)"}]
```

Out[93]=



```
mlist = {};
Monitor[
  Do[
    Clear[t];
    r = r0;
    tab = {0.25};
    rliste = {r};
    pliste = {};
    driver = (R - First[R]) / (Last[R] - First[R]);
    Monitor[
     Do[
       rand = RandomReal[NormalDistribution[0, σ], 5 * 31];
       rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
       ifun = Interpolation[rand];
       (* xxxxxxxx *)
       r = r0 + v * driver[[30 * t]];
       rliste = Append[rliste, r];
       Q0 = Last[tab];
       s = NDSolve[{Q'[tt] == a * (r - Sqrt[(1 - Q[tt])^2] * Q[tt]) + ifun[tt + 1],
          Q[0] == Q0}, Q, {tt, 0, 30}];
       mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
       tab = Join[tab, mid];
       , {t, 1, 316}];
     , t];
    icemodel = -tab;
    model = A + B * (icemodel);
    mlist = Append[mlist, model];
    , {j, 1, 20}];
  , j];
```

```
mean = Map[Mean[#] &, Transpose[mlist]];
low = Map[Quantile[#, 0.025] &, Transpose[mlist]];
high = Map[Quantile[#, 1 - 0.025] &, Transpose[mlist]];


Qmean = ListPlot[mean, Joined → True,
   PlotRange → All, PlotStyle → {Red, Opacity[0.5]}];
Qampl = ListPlot[{low, high}, Joined → True, PlotRange → All,
   PlotStyle → {Pink}, Filling → {1 → {2}}];
Q11 = ListPlot[xx, Frame → True, FrameTicks →
     {{Automatic, Automatic}, {ticks, None}}, FrameStyle → Directive[16],
   Joined → True, FrameLabel → {"year", "sea ice extent (10^6 km)"}];
Show[{Q11, Qmean, Qampl}, PlotRange → All]
```

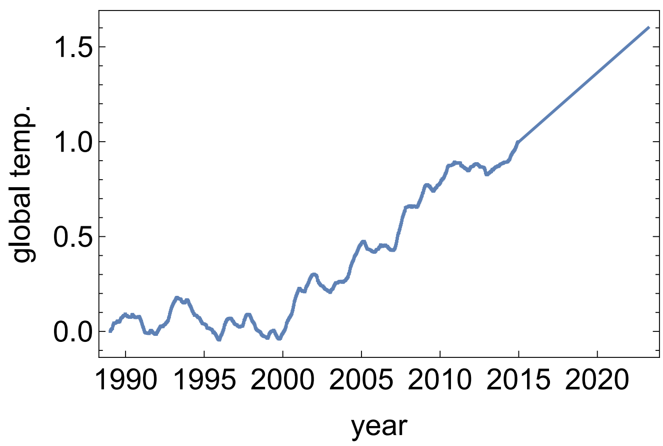


Future prediction

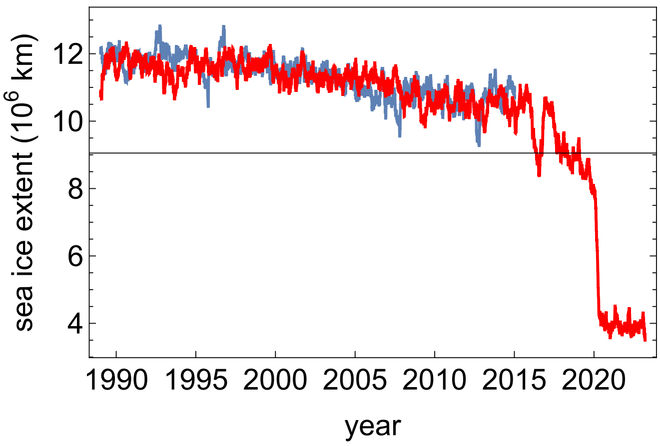
Scenario for temperature:

```
In[94]:= ticks = {{359, "1990"}, {359 + 5 * 365, "1995"},
    {359 + 10 * 365, "2000"}, {359 + 15 * 365, "2005"}, {359 + 20 * 365, "2010"},
    {359 + 25 * 365, "2015"}, {359 + 30 * 365, "2020"}};
newdriver = Join[driver, Last[driver] + 0.0002 * Range[3000]];
ListPlot[newdriver, Frame → True,
  FrameTicks → {{Automatic, Automatic}, {ticks, None}},
  FrameStyle → Directive[16], Joined → True,
  FrameLabel → {"year", "global temp."}]
```

Out[96]=

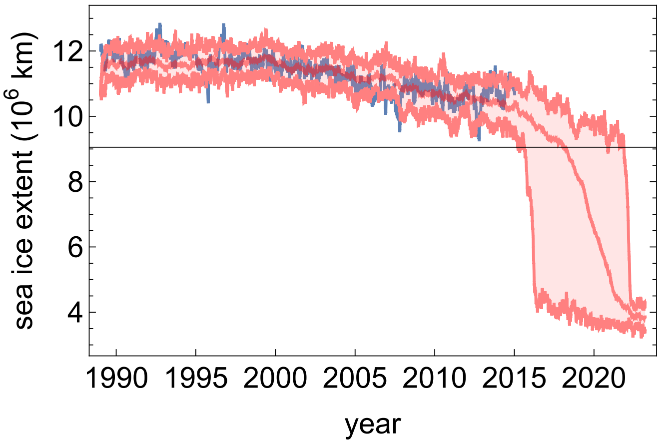


```
Clear[t];
a = 0.02752;
Clear[t];
σ = 0.017;
r0 = 0.1;
v = 0.12;
A = 12.5;
B = 7;
r = r0;
tab = {0.25};
rliste = {r};
pliste = {};
driver = (R - First[R]) / (Last[R] - First[R]);
Monitor[
 Do[
   rand = RandomReal[NormalDistribution[0, σ], 5 * 31];
   rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
   ifun = Interpolation[rand];
   (* xxxxxxxx *)
   r = r0 + v * newdriver[[30 * t]];
   rliste = Append[rliste, r];
   Q0 = Last[tab];
   s = NDSolve[{Q'[tt] == a * (r - Sqrt[(1 - Q[tt]) ^ 2] * Q[tt]) + ifun[tt + 1],
      Q[0] == Q0}, Q, {tt, 0, 30}];
   mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
   tab = Join[tab, mid];
   , {t, 1, 316 + 100}];
 , t]
icemodel = -tab;
Q3 =
  ListPlot[A + B * (icemodel), Joined → True, PlotRange → All, PlotStyle → Red];

Show[{Q1, Q3}, PlotRange → All,
 FrameTicks → {{Automatic, Automatic}, {ticks, None}}]
```



```
mlist = {};
Monitor[
  Do[
    Clear[t];
    r = r0;
    tab = {0.25};
    rliste = {r};
    pliste = {};
    driver = (R - First[R]) / (Last[R] - First[R]);
    Monitor[
     Do[
       rand = RandomReal[NormalDistribution[0, σ], 5 * 31];
       rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
       ifun = Interpolation[rand];
       (* xxxxxxxx *)
       r = r0 + v * newdriver[[30 * t]];
       rliste = Append[rliste, r];
       Q0 = Last[tab];
       s = NDSolve[{Q'[tt] == a * (r - Sqrt[(1 - Q[tt])^2] * Q[tt]) + ifun[tt + 1],
           Q[0] == Q0}, Q, {tt, 0, 30}];
       mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
       tab = Join[tab, mid];
       , {t, 1, 316 + 100}];
     , t];
    icemodel = -tab;
    model = A + B * (icemodel);
    mlist = Append[mlist, model];
    , {j, 1, 20}];
  , j];
```

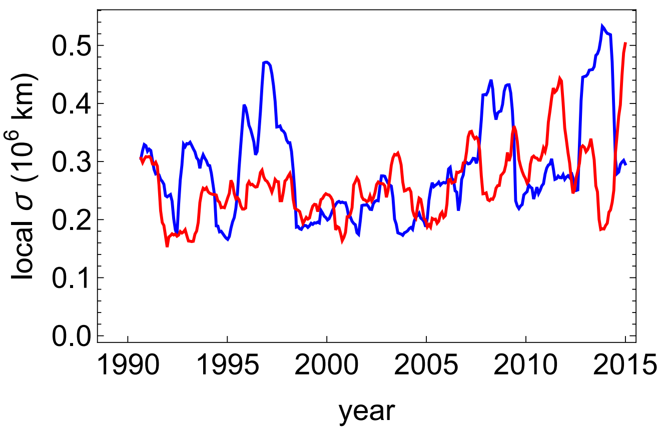
```
mean = Map[Mean[#] &, Transpose[mlist]];
low = Map[Quantile[#, 0.025] &, Transpose[mlist]];
high = Map[Quantile[#, 1 - 0.025] &, Transpose[mlist]];


Qmean = ListPlot[mean, Joined → True,
   PlotRange → All, PlotStyle → {Red, Opacity[0.5]}];
Qampl = ListPlot[{low, high}, Joined → True, PlotRange → All,
   PlotStyle → {Pink}, Filling → {1 → {2}}];
Q11 = ListPlot[xx, Frame → True, FrameTicks →
    {{Automatic, Automatic}, {ticks, None}}, FrameStyle → Directive[16],
   Joined → True, FrameLabel → {"year", "sea ice extent (10^6 km)"}];
Show[{Q11, Qmean, Qampl}, PlotRange → All]
```



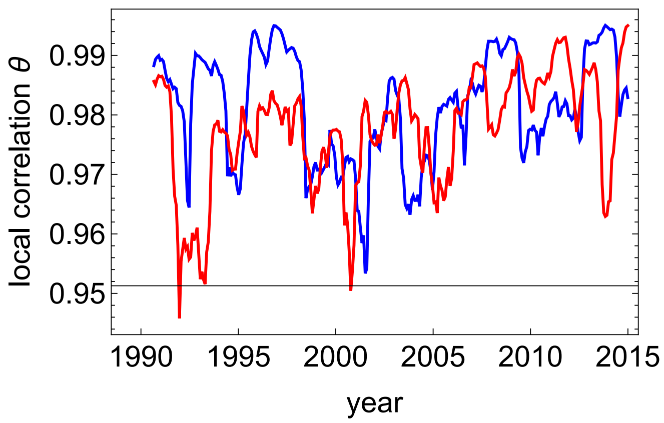
Early warning indications

Estimate variance and correlation. Red for model. Blue for real data.

```
win = 600;
σlist = {};
τlist = {};
Monitor[
  Do[
    qqq = modelexample[[t - win ;; t]];
    σloc = StandardDeviation[qqq]; (*local standard deviation*)
    τloc = Correlation[Drop[qqq, 1], Drop[qqq, -1]];
    (*local correlation scale*)
    σlist = Append[σlist, {t, σloc}];
    τlist = Append[τlist, {t, τloc}];
    , {t, win + 1, Length[modelexample], 30}];
  , t];
QQ1 = ListPlot[σlist, Frame → True,
   FrameTicks → {{Automatic, Automatic}, {ticks, None}},
   FrameStyle → Directive[16], Joined → True,
   FrameLabel → {"year", "local σ (10^6 km)"}, PlotStyle → Red];
QQ2 = ListPlot[τlist, Frame → True, FrameTicks →
    {{Automatic, Automatic}, {ticks, None}}, FrameStyle → Directive[16],
   Joined → True, FrameLabel → {"year", "local correlation θ"}, PlotStyle → Red];
win = 600;
σlist = {};
τlist = {};
Monitor[
  Do[
    qqq = xx[[t - win ;; t]];
    σloc = StandardDeviation[qqq]; (*local standard deviation*)
    τloc = Correlation[Drop[qqq, 1], Drop[qqq, -1]];
    (*local correlation scale*)
    σlist = Append[σlist, {t, σloc}];
    τlist = Append[τlist, {t, τloc}];
    , {t, win + 1, Length[modelexample], 30}];
  , t];
QQ3 = ListPlot[σlist, Frame → True,
   FrameTicks → {{Automatic, Automatic}, {ticks, None}},
   FrameStyle → Directive[16], Joined → True,
   FrameLabel → {"year", "local σ (10^6 km)"}, PlotStyle → Blue];
QQ4 = ListPlot[τlist, Frame → True,
   FrameTicks → {{Automatic, Automatic}, {ticks, None}},
   FrameStyle → Directive[16], Joined → True,
   FrameLabel → {"year", "local correlation θ"}, PlotStyle → Blue];
Show[{QQ3, QQ1}, PlotRange → All]
```



```
Show[{QQ4, QQ2}, PlotRange → All]
```

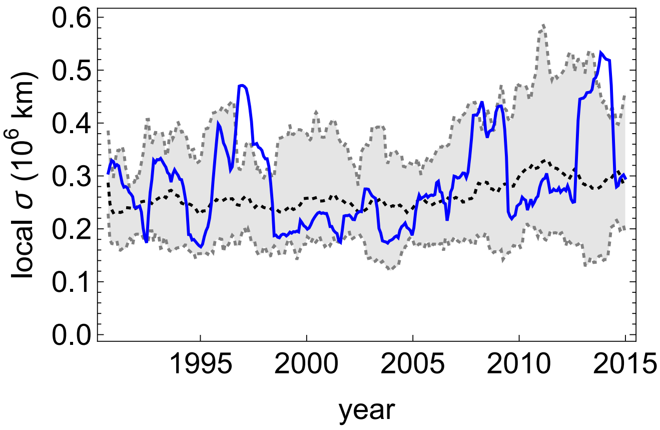


Monte Carlo

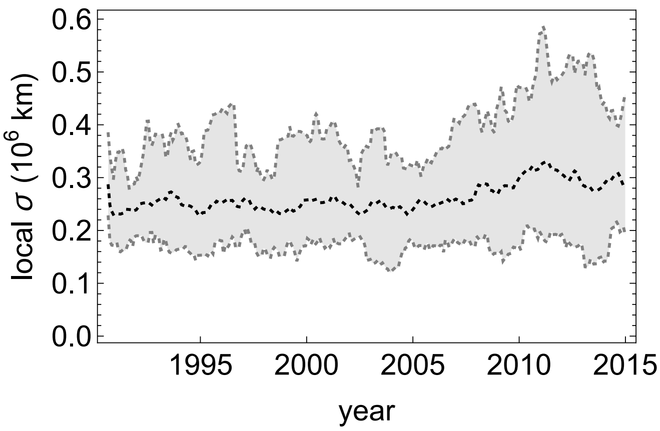
```
σlistlist = {};
τlistlist = {};
```

```
Monitor[
 Do[
   Clear[t];
   r = r0;
   tab = {0.25};
   rliste = {r};
   pliste = {};
   driver = (R - First[R]) / (Last[R] - First[R]);
   Monitor[
     Do[
       rand = RandomReal[NormalDistribution[0, σ], 5 * 31];
       rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
       ifun = Interpolation[rand];
       (* xxxxxxxx *)
       r = r0 + v * driver[[30 * t]];
       rliste = Append[rliste, r];
       Q0 = Last[tab];
       s = NDSolve[{Q'[tt] == a * (r - Sqrt[(1 - Q[tt])^2] * Q[tt]) + ifun[tt + 1],
           Q[0] == Q0}, Q, {tt, 0, 30}];
       mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
       tab = Join[tab, mid];
       , {t, 1, 316}];
     , t]
    icemodel = -tab;
   modelexample = A + B * (icemodel);
   win = 600;
   σlist = {};
   τlist = {};
   Do[
    qqq = modelexample[[t - win ;; t]];
    σloc = StandardDeviation[qqq]; (*local standard deviation*)
    τloc = Correlation[Drop[qqq, 1], Drop[qqq, -1]];
    (*local correlation scale*)
    σlist = Append[σlist, {t, σloc}];
    τlist = Append[τlist, {t, τloc}];
    , {t, win + 1, Length[modelexample], 30}];
   σlistlist = Append[σlistlist, σlist];
   τlistlist = Append[τlistlist, τlist];
   , {run, 1, 20}];
  , {run, t}]
```

```
times = σlistlist[[1]][[All, 1]];
ticks = {{359, "1990"}, {359 + 5 * 365, "1995"},
   {359 + 10 * 365, "2000"}, {359 + 15 * 365, "2005"}, {359 + 20 * 365, "2010"},
   {359 + 25 * 365, "2015"}, {359 + 30 * 365, "2020"}};
σmean = Map[Mean[#] &, Transpose[Map[#[[All, 2]] &, σlistlist]]];
σlow = Map[Quantile[#, 0.025] &, Transpose[Map[#[[All, 2]] &, σlistlist]]];
σhigh = Map[Quantile[#, 1 - 0.025] &, Transpose[Map[#[[All, 2]] &, σlistlist]]];

σmean = Thread[{#1, #2} &[times, σmean]];
σlow = Thread[{#1, #2} &[times, σlow]];
σhigh = Thread[{#1, #2} &[times, σhigh]];

PL1 = ListPlot[σmean, Frame → True,
   FrameTicks → {{Automatic, Automatic}, {ticks, None}},
   FrameStyle → Directive[16], Joined → True,
   FrameLabel → {"year", "local correlation θ"}, PlotStyle → {Black, Dotted}];
PL2 = ListPlot[{σlow, σhigh}, Frame → True, FrameTicks →
    {{Automatic, Automatic}, {ticks, None}}, FrameStyle → Directive[16],
   Joined → True, FrameLabel → {"year", "local σ (10^6 km)"},
   PlotStyle → {{Dotted, Gray}, {Dotted, Gray}},
   Filling → {1 → {2}}, PlotRange → All];
PL3 = Show[PL2, PL1, QQ3]
```




```
Show[PL2, PL1]
```



Appendix E

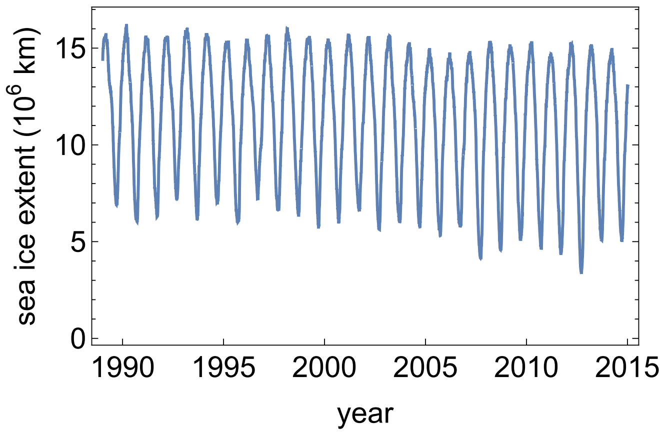
Global temperature and Brownian motion

The following code contains all the calculations and plots for real data and modeling of future predictions of sea ice extent in the Arctic using global temperature as a driver and Brownian motion as an additive noise, discussed in section "Brownian motion" in the chapter "Global temperature". The code was written in the program "Mathematica".

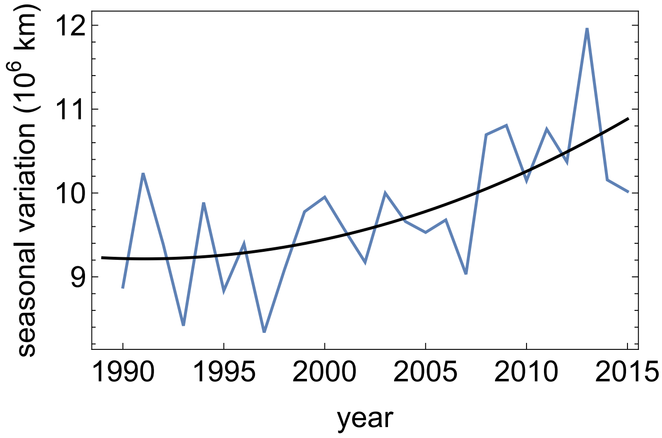
Reading and preparing data:

Read sea ice data:

```
x = ReadList[
    "Dropbox/Master Theses/sea ice/NH_seaice_extent_final.csv", String];
x = Table[ToExpression[StringSplit[x[[t]], ","][[4]]], {t, 3, Length[x]}];
ticks = {{359, "1990"}, {359 + 5 * 365, "1995"}, {359 + 10 * 365, "2000"},
    {359 + 15 * 365, "2005"}, {359 + 20 * 365, "2010"}, {359 + 25 * 365, "2015"}};
y = Drop[x, 2096];
ListPlot[y, Frame → True, FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "sea ice extent (10^6 km)"}]
```

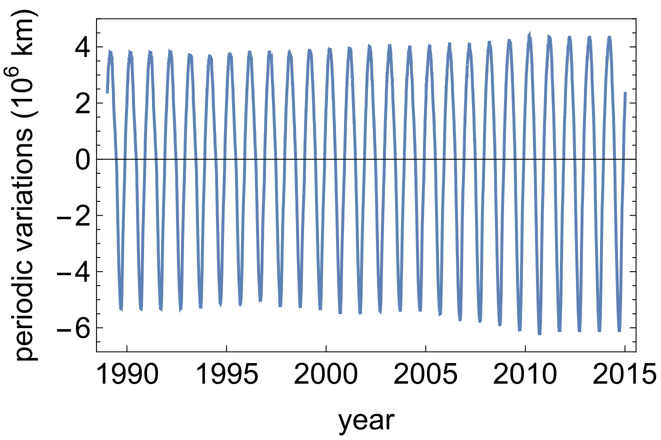



```
amplitudes = Map[Max[#] - Min[#] &, Partition[y, 365]];
fit = Fit[amplitudes, {zz^2, zz, 1}, zz];
PL1 = ListPlot[amplitudes, Joined → True];
PL2 = Plot[fit, {zz, 0, 26}, PlotStyle → Black];
newticks = {{1 + 0, "1990"}, {1 + 5, "1995"},
    {1 + 10, "2000"}, {1 + 15, "2005"}, {1 + 20, "2010"}, {1 + 25, "2015"}};
Show[{PL1, PL2}, Frame → True, FrameTicks →
   {{Automatic, Automatic}, {newticks, None}}, FrameStyle → Directive[16],
 FrameLabel → {"year", "seasonal variation (10^6 km)"}]
```



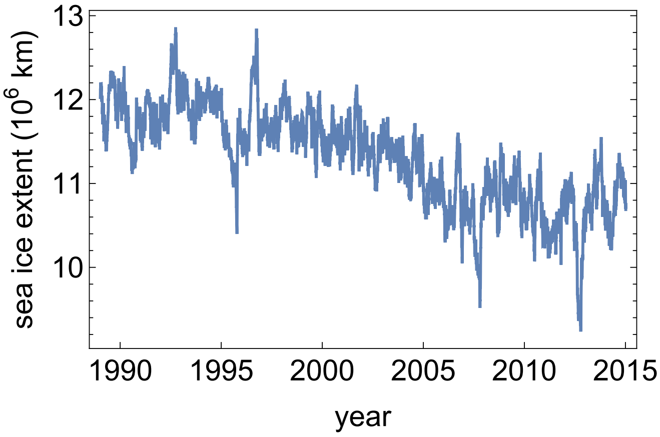
```
clim1 = Map[Mean[#] &, Transpose[Partition[y, 365][[1 ;; 5]]]];
clim2 = Map[Mean[#] &, Transpose[Partition[y, 365][[1 ;; 5]]]];
clim3 = Map[Mean[#] &, Transpose[Partition[y, 365][[1 ;; 5]]]];
clim4 = Map[Mean[#] &, Transpose[Partition[y, 365][[2 ;; 6]]]];
clim5 = Map[Mean[#] &, Transpose[Partition[y, 365][[3 ;; 7]]]];
clim6 = Map[Mean[#] &, Transpose[Partition[y, 365][[4 ;; 8]]]];
clim7 = Map[Mean[#] &, Transpose[Partition[y, 365][[5 ;; 9]]]];
clim8 = Map[Mean[#] &, Transpose[Partition[y, 365][[6 ;; 10]]]];
clim9 = Map[Mean[#] &, Transpose[Partition[y, 365][[7 ;; 11]]]];
clim10 = Map[Mean[#] &, Transpose[Partition[y, 365][[8 ;; 12]]]];
clim11 = Map[Mean[#] &, Transpose[Partition[y, 365][[9 ;; 13]]]];
clim12 = Map[Mean[#] &, Transpose[Partition[y, 365][[10 ;; 14]]]];
clim13 = Map[Mean[#] &, Transpose[Partition[y, 365][[11 ;; 15]]]];
clim14 = Map[Mean[#] &, Transpose[Partition[y, 365][[12 ;; 16]]]];
clim15 = Map[Mean[#] &, Transpose[Partition[y, 365][[13 ;; 17]]]];
clim16 = Map[Mean[#] &, Transpose[Partition[y, 365][[14 ;; 18]]]];
clim17 = Map[Mean[#] &, Transpose[Partition[y, 365][[15 ;; 19]]]];
clim18 = Map[Mean[#] &, Transpose[Partition[y, 365][[16 ;; 20]]]];
clim19 = Map[Mean[#] &, Transpose[Partition[y, 365][[17 ;; 21]]]];
clim20 = Map[Mean[#] &, Transpose[Partition[y, 365][[18 ;; 22]]]];
clim21 = Map[Mean[#] &, Transpose[Partition[y, 365][[19 ;; 23]]]];
clim22 = Map[Mean[#] &, Transpose[Partition[y, 365][[20 ;; 24]]]];
clim23 = Map[Mean[#] &, Transpose[Partition[y, 365][[21 ;; 25]]]];
clim24 = Map[Mean[#] &, Transpose[Partition[y, 365][[22 ;; 26]]]];
clim25 = Map[Mean[#] &, Transpose[Partition[y, 365][[22 ;; 26]]]];
clim26 = Map[Mean[#] &, Transpose[Partition[y, 365][[22 ;; 26]]]];
climtab = {clim1, clim2, clim3, clim4, clim5, clim6, clim7, clim8, clim9,
   clim10, clim11, clim12, clim13, clim14, clim15, clim16, clim17, clim18,
   clim19, clim20, clim21, clim22, clim23, clim24, clim25, clim26};
climtab = Map[# - Mean[#] &, climtab];
factors = Map[Max[#] - Min[#] &, climtab];
climseries = Flatten[climtab];
```

```
ListPlot[climseries, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "periodic variations (10^6 km)"}]
```



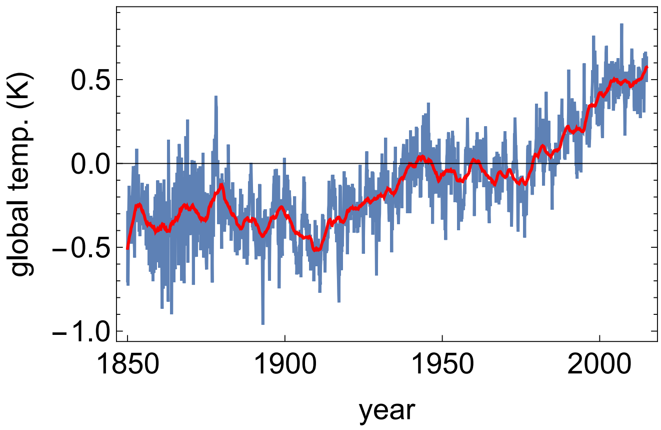
Remove climatology:

```
xx = y - climseries;
years = Partition[xx, 365];
yearmeans = Map[Mean[#] &, years];
years = Table[(years[[i]] - yearmeans[[i]]) * (factors[[i]] / Mean[factors]) +
    yearmeans[[i]], {i, 1, 26}];
xx = Flatten[years];
Q1 = ListPlot[xx, Frame → True,
  FrameTicks → {{Automatic, Automatic}, {ticks, None}},
  FrameStyle → Directive[16], Joined → True,
  FrameLabel → {"year", "sea ice extent (10^6 km)"}]
```

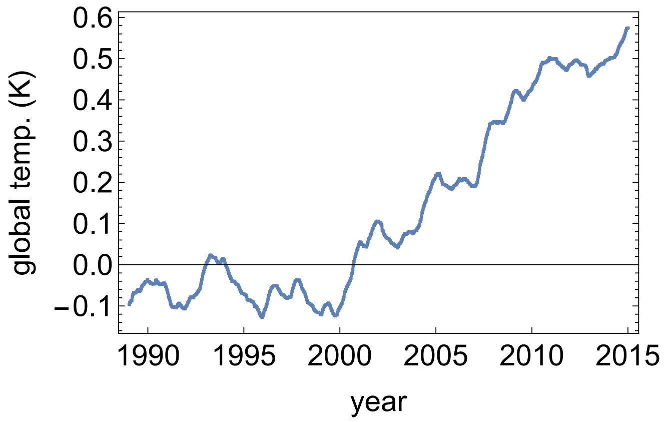


Global temperature anomaly:

```
T = ToExpression[
   StringSplit[Drop[ReadList["Dropbox/Master Theses/sea ice/globaltemp.txt",
       String], -1]][[All, 2]]];
ticks2 = {{1, "1850"}, {1 + 12 * 50, "1900"}, {1 + 12 * 100, "1950"},
   {1 + 12 * 150, "2000"}};
PL1 = ListPlot[T, Frame → True, FrameTicks →
    {{Automatic, Automatic}, {ticks2, None}}, FrameStyle → Directive[16],
   Joined → True, FrameLabel → {"year", "global temp. (K)"}];
av = Drop[MovingAverage[ArrayPad[T, 5 * 12 / 2, "Fixed"], 5 * 12], -1];
PL2 = ListPlot[av, PlotStyle → Red, Joined → True];
Show[PL1, PL2]
```



```
R = Flatten[Table[Table[av[[t]], {12}], {t, 1, Length[av]}]];
R = R[[Length[R] - Length[xx] + 1 ;; Length[R]]];
ListPlot[R, Frame → True, FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "global temp. (K)"}]
```



Modeling

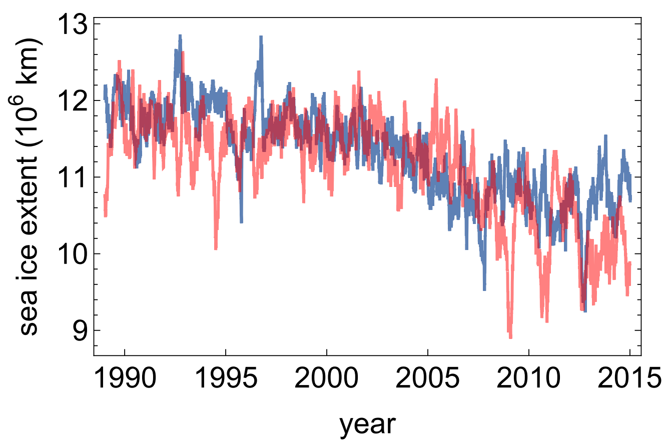
```
OU = xx[[3000 ;; 6000]];
ΔOU = Drop[OU, 1] - Drop[OU, -1];
ΔOU = ΔOU - Mean[ΔOU];
OU = FoldList[Plus, 0, ΔOU];
OU = OU - Mean[OU];
OU = Thread[{#1, #2} &[Range[Length[OU]], OU]];
est = EstimatedProcess[OU, OrnsteinUhlenbeckProcess[μ, Σ, θ]]
```

```
OrnsteinUhlenbeckProcess[0.0000320507, 0.26038, 0.0182579]
```

```
(*0.14 is the mean r-value*)
Clear[a]
sol = Solve[a Sqrt[1 - 4 * 0.14] == est[[3]], a]
```

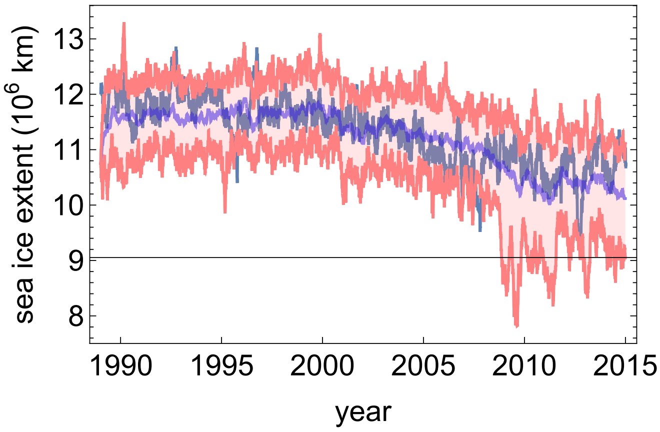
```
{{a → 0.0275248}}
```

```
a = 0.02752;
Clear[t];
σ = 0.012;
r0 = 0.1;
v = 0.12;
A = 12.5;
B = 7;
r = r0;
tab = {0.25};
rliste = {r};
pliste = {};
driver = (R - First[R]) / (Last[R] - First[R]);
Monitor[
 Do[
   rand =
    RandomFunction[FractionalBrownianMotionProcess[.64], {0, 5 * 31 + 1, 1}];
   rand = Drop[rand["Path"][[All, 2]], 1];
   rand = Drop[rand, 1] - Drop[rand, -1];
   rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
   ifun = Interpolation[rand];
   (* xxxxxxxx *)
   r = r0 + v * driver[[30 * t]];
   rliste = Append[rliste, r];
   Q0 = Last[tab];
   s = NDSolve[{Q'[tt] == a * (r - Sqrt[(1 - Q[tt]) ^2] * Q[tt]) + σ * ifun[tt + 1],
      Q[0] == Q0}, Q, {tt, 0, 30}];
   mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
   tab = Join[tab, mid];
   , {t, 1, 316}];
 , t]
icemodel = -tab;

Q2 = ListPlot[A + B * (icemodel), Joined → True,
   PlotRange → All, PlotStyle → {Red, Opacity[0.5]}];
modelexample = A + B * (icemodel);
Show[{Q1, Q2}, PlotRange → All]
```

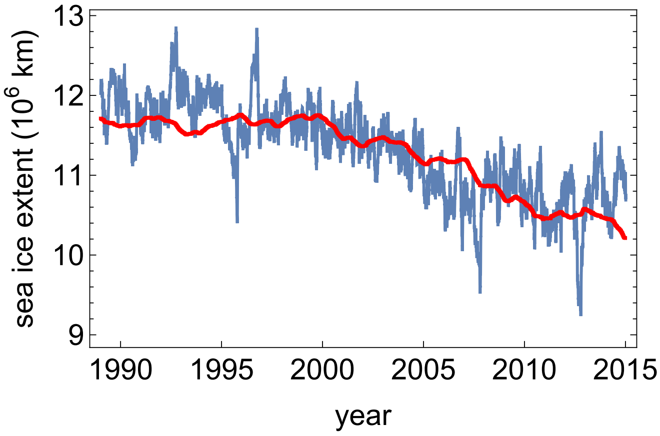


```
mlist = {};
Monitor[
  Do[
    Clear[t];
    r = r0;
    tab = {0.25};
    rliste = {r};
    pliste = {};
    driver = (R - First[R]) / (Last[R] - First[R]);
    Monitor[
     Do[
       rand =
        RandomFunction[FractionalBrownianMotionProcess[.64], {0, 5 * 31 + 1, 1}];
       rand = Drop[rand["Path"][[All, 2]], 1];
       rand = Drop[rand, 1] - Drop[rand, -1];
       rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
       ifun = Interpolation[rand];
       (* xxxxxxxx *)
       r = r0 + v * driver[[30 * t]];
       rliste = Append[rliste, r];
       Q0 = Last[tab];
       s = NDSolve[{Q'[tt] == a * (r - Sqrt[(1 - Q[tt])^2] * Q[tt]) + σ * ifun[tt + 1],
           Q[0] == Q0}, Q, {tt, 0, 30}];
       mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
       tab = Join[tab, mid];
       , {t, 1, 316}];
     , t];
    icemodel = -tab;
    model = A + B * (icemodel);
    mlist = Append[mlist, model];
    , {j, 1, 20}];
  , j];
```

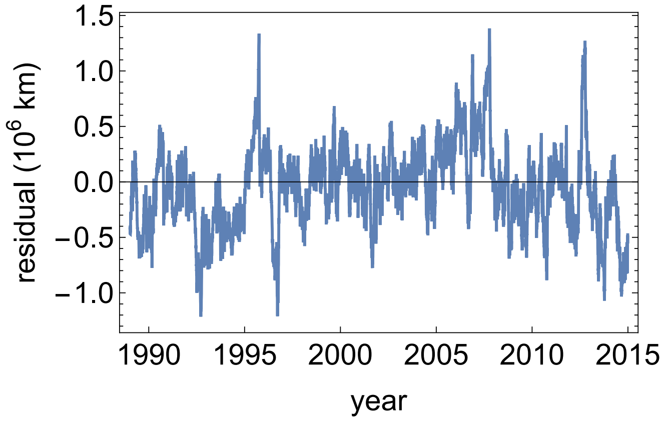
```
mean = Map[Mean[#] &, Transpose[mlist]];
low = Map[Quantile[#, 0.025] &, Transpose[mlist]];
high = Map[Quantile[#, 1 - 0.025] &, Transpose[mlist]];


Qmean = ListPlot[mean, Joined → True,
   PlotRange → All, PlotStyle → {Blue, Opacity[0.5]}];
Qampl = ListPlot[{low, high}, Joined → True, PlotRange → All,
   PlotStyle → {Pink}, Filling → {1 → {2}}];
Q11 = ListPlot[xx, Frame → True, FrameTicks →
    {{Automatic, Automatic}, {ticks, None}}, FrameStyle → Directive[16],
   Joined → True, FrameLabel → {"year", "sea ice extent (10^6 km)"}];
Show[{Q11, Qmean, Qampl}, PlotRange → All]
```




```
Clear[r]
r0 = 0.1;
v = 0.12;
A = 12.5;
B = 7;
fix[r_] := 1/2 (1 - √(1 - 4 r));
Q3 = ListPlot[A - B * Map[fix[#] &, r0 + v * driver[[Range[30 * 316]]]],
   PlotStyle → {Red, Thick}, Joined → True];
Show[{Q1, Q3}, PlotRange -> All]
```



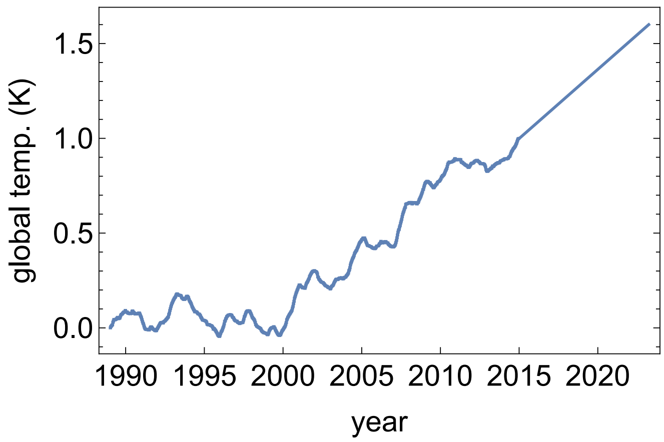
```
residual = (A - B * Map[fix[#] &, r0 + v * driver[[Range[30 * 316]]]]) - Drop[xx, 10];
ListPlot[residual, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "residual (10^6 km)"}]
```



Future prediction

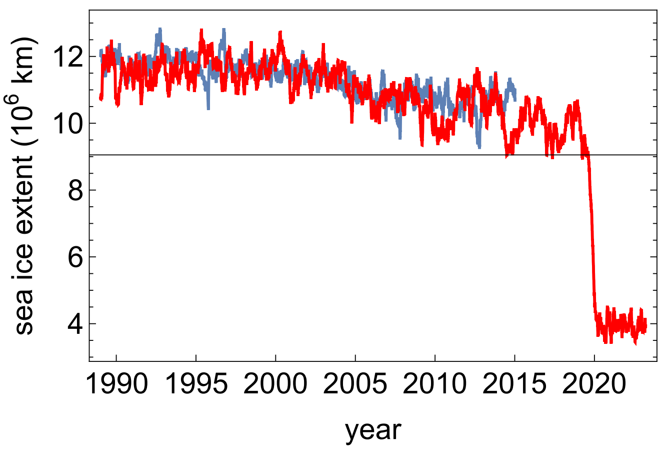
Scenario for temperature:

```
ticks = {{359, "1990"}, {359 + 5 * 365, "1995"},
   {359 + 10 * 365, "2000"}, {359 + 15 * 365, "2005"}, {359 + 20 * 365, "2010"},
   {359 + 25 * 365, "2015"}, {359 + 30 * 365, "2020"}};
newdriver = Join[driver, Last[driver] + 0.0002 * Range[3000]];
ListPlot[newdriver, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "global temp. (K)"}]
```



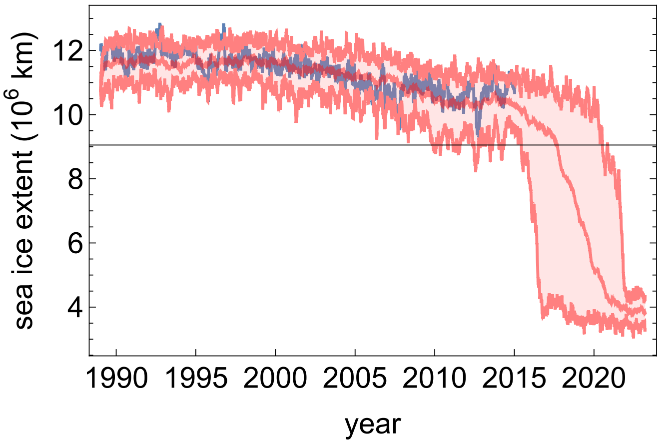
```
Clear[t];
r = r0;
tab = {0.25};
rliste = {r};
pliste = {};
driver = (R - First[R]) / (Last[R] - First[R]);
Monitor[
 Do[
   rand =
    RandomFunction[FractionalBrownianMotionProcess[.64], {0, 5 * 31 + 1, 1}];
   rand = Drop[rand["Path"][[All, 2]], 1];
   rand = Drop[rand, 1] - Drop[rand, -1];
   rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
   ifun = Interpolation[rand];
   (* xxxxxxxx *)
   r = r0 + v * newdriver[[30 * t]];
   rliste = Append[rliste, r];
   Q0 = Last[tab];
   s = NDSolve[{Q'[tt] == a * (r - Sqrt[(1 - Q[tt])^2] * Q[tt]) + σ * ifun[tt + 1],
      Q[0] == Q0}, Q, {tt, 0, 30}];
   mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
   tab = Join[tab, mid];
   , {t, 1, 316 + 100}];
 , t]
icemodel = -tab;
Q3 =
  ListPlot[A + B * (icemodel), Joined → True, PlotRange → All, PlotStyle → {Red}];
```

```
Show[{Q1, Q3}, PlotRange → All,
 FrameTicks → {{Automatic, Automatic}, {ticks, None}}]
```



```
mlist = {};
Monitor[
  Do[
    Clear[t];
    r = r0;
    tab = {0.25};
    rliste = {r};
    pliste = {};
    driver = (R - First[R]) / (Last[R] - First[R]);
    Monitor[
     Do[
       rand =
        RandomFunction[FractionalBrownianMotionProcess[.64], {0, 5 * 31 + 1, 1}];
       rand = Drop[rand["Path"][[All, 2]], 1];
       rand = Drop[rand, 1] - Drop[rand, -1];
       rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
       ifun = Interpolation[rand];
       (* xxxxxxxx *)
       r = r0 + v * newdriver[[30 * t]];
       rliste = Append[rliste, r];
       Q0 = Last[tab];
       s = NDSolve[{Q'[tt] == a * (r - Sqrt[(1 - Q[tt])^2] * Q[tt]) + σ * ifun[tt + 1],
          Q[0] == Q0}, Q, {tt, 0, 30}];
       mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
       tab = Join[tab, mid];
       , {t, 1, 316 + 100}];
     , t];
    icemodel = -tab;
    model = A + B * (icemodel);
    mlist = Append[mlist, model];
    , {j, 1, 20}];
  , j];
```

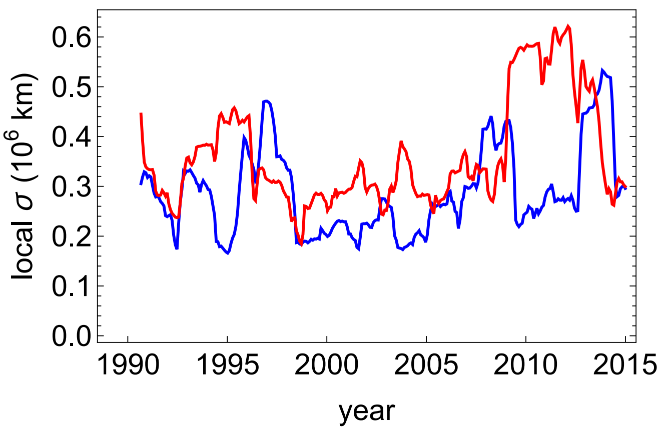
```
mean = Map[Mean[#] &, Transpose[mlist]];
low = Map[Quantile[#, 0.025] &, Transpose[mlist]];
high = Map[Quantile[#, 1 - 0.025] &, Transpose[mlist]];


Qmean = ListPlot[mean, Joined → True,
   PlotRange → All, PlotStyle → {Red, Opacity[0.5]}];
Qampl = ListPlot[{low, high}, Joined → True, PlotRange → All,
   PlotStyle → {Pink}, Filling → {1 → {2}}];
Q11 = ListPlot[xx, Frame → True, FrameTicks →
     {{Automatic, Automatic}, {ticks, None}}, FrameStyle → Directive[16],
   Joined → True, FrameLabel → {"year", "sea ice extent (10^6 km)"}];
Show[{Q11, Qmean, Qampl}, PlotRange → All]
```



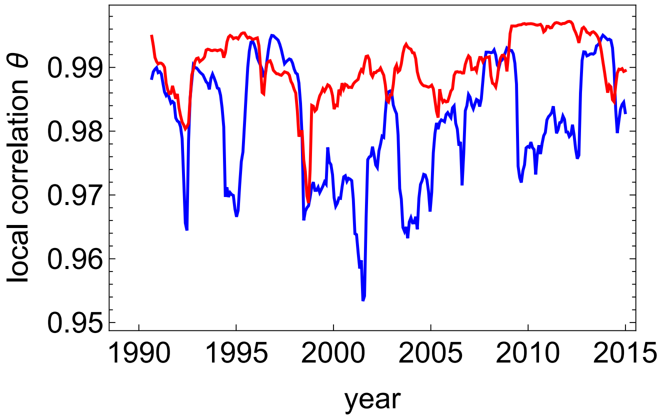
Early warning indications

Estimate variance and correlation. Red for model. Blue for real data.

```
win = 600;
σlist = {};
τlist = {};
Monitor[
  Do[
    qqq = modelexample[[t - win ;; t]];
    σloc = StandardDeviation[qqq]; (*local standard deviation*)
    τloc = Correlation[Drop[qqq, 1], Drop[qqq, -1]];
    (*local correlation scale*)
    σlist = Append[σlist, {t, σloc}];
    τlist = Append[τlist, {t, τloc}];
    , {t, win + 1, Length[modelexample], 30}];
  , t];
QQ1 = ListPlot[σlist, Frame → True,
   FrameTicks → {{Automatic, Automatic}, {ticks, None}},
   FrameStyle → Directive[16], Joined → True,
   FrameLabel → {"year", "local σ (10^6 km)"}, PlotStyle → Red];
QQ2 = ListPlot[τlist, Frame → True, FrameTicks →
    {{Automatic, Automatic}, {ticks, None}}, FrameStyle → Directive[16],
   Joined → True, FrameLabel → {"year", "local correlation θ"}, PlotStyle → Red];
win = 600;
σlist = {};
τlist = {};
Monitor[
  Do[
    qqq = xx[[t - win ;; t]];
    σloc = StandardDeviation[qqq]; (*local standard deviation*)
    τloc = Correlation[Drop[qqq, 1], Drop[qqq, -1]];
    (*local correlation scale*)
    σlist = Append[σlist, {t, σloc}];
    τlist = Append[τlist, {t, τloc}];
    , {t, win + 1, Length[modelexample], 30}];
  , t];
QQ3 = ListPlot[σlist, Frame → True,
   FrameTicks → {{Automatic, Automatic}, {ticks, None}},
   FrameStyle → Directive[16], Joined → True,
   FrameLabel → {"year", "local σ (10^6 km)"}, PlotStyle → Blue];
QQ4 = ListPlot[τlist, Frame → True,
   FrameTicks → {{Automatic, Automatic}, {ticks, None}},
   FrameStyle → Directive[16], Joined → True,
   FrameLabel → {"year", "local correlation θ"}, PlotStyle → Blue];
Show[{QQ3, QQ1}, PlotRange → All]
```



```
Show[{QQ4, QQ2}, PlotRange → All]
```

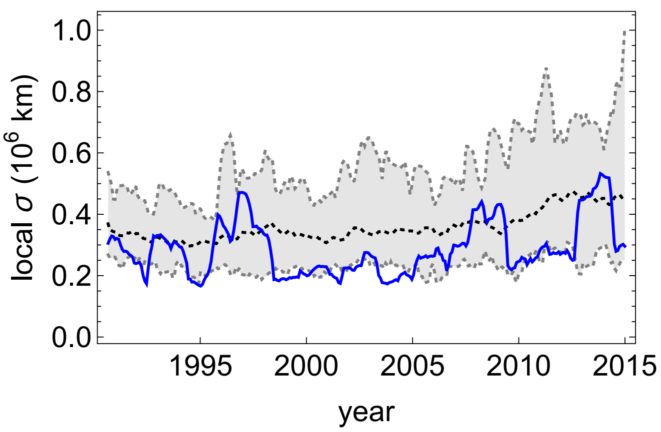


Monte Carlo

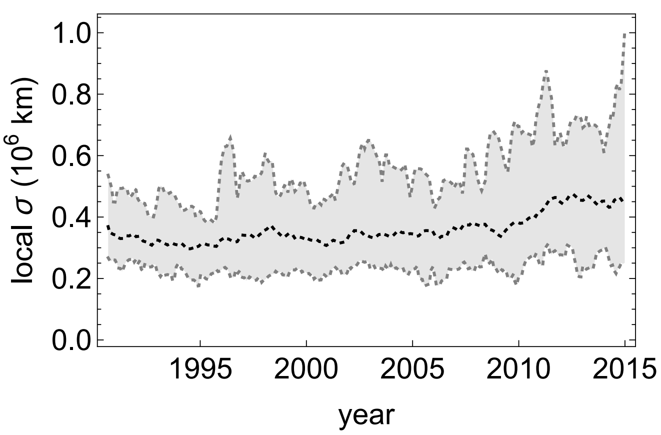
```
σlistlist = {};
τlistlist = {};
```

```
Monitor[
 Do[
   Clear[t];
   r = r0;
   tab = {0.25};
   rliste = {r};
   pliste = {};
   driver = (R - First[R]) / (Last[R] - First[R]);
   Monitor[
     Do[
       rand =
        RandomFunction[FractionalBrownianMotionProcess[.64], {0, 5 * 31 + 1, 1}];
       rand = Drop[rand["Path"][[All, 2]], 1];
       rand = Drop[rand, 1] - Drop[rand, -1];
       rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
       ifun = Interpolation[rand];
       (* xxxxxxxx *)
       r = r0 + v * driver[[30 * t]];
       rliste = Append[rliste, r];
       Q0 = Last[tab];
       s = NDSolve[{Q'[tt] == a * (r - Sqrt[(1 - Q[tt])^2] * Q[tt]) + σ * ifun[tt + 1],
           Q[0] == Q0}, Q, {tt, 0, 30}];
       mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
       tab = Join[tab, mid];
       , {t, 1, 316}];
     , t]
    icemodel = -tab;
   modelexample = A + B * (icemodel);
   win = 600;
   σlist = {};
   τlist = {};
   Do[
    qqq = modelexample[[t - win ;; t]];
    σloc = StandardDeviation[qqq]; (*local standard deviation*)
    τloc = Correlation[Drop[qqq, 1], Drop[qqq, -1]];
    (*local correlation scale*)
    σlist = Append[σlist, {t, σloc}];
    τlist = Append[τlist, {t, τloc}];
    , {t, win + 1, Length[modelexample], 30}];
   σlistlist = Append[σlistlist, σlist];
   τlistlist = Append[τlistlist, τlist];
   , {run, 1, 20}];
 , {run, t}]
```

```
times = σlistlist[[1]][[All, 1]];
ticks = {{359, "1990"}, {359 + 5 * 365, "1995"},
   {359 + 10 * 365, "2000"}, {359 + 15 * 365, "2005"}, {359 + 20 * 365, "2010"},
   {359 + 25 * 365, "2015"}, {359 + 30 * 365, "2020"}};
σmean = Map[Mean[#] &, Transpose[Map[#[[All, 2]] &, σlistlist]]];
σlow = Map[Quantile[#, 0.025] &, Transpose[Map[#[[All, 2]] &, σlistlist]]];
σhigh = Map[Quantile[#, 1 - 0.025] &, Transpose[Map[#[[All, 2]] &, σlistlist]]];

σmean = Thread[{#1, #2} &[times, σmean]];
σlow = Thread[{#1, #2} &[times, σlow]];
σhigh = Thread[{#1, #2} &[times, σhigh]];

PL1 = ListPlot[σmean, Frame → True,
   FrameTicks → {{Automatic, Automatic}, {ticks, None}},
   FrameStyle → Directive[16], Joined → True,
   FrameLabel → {"year", "local correlation θ"}, PlotStyle → {Black, Dotted}];
PL2 = ListPlot[{σlow, σhigh}, Frame → True, FrameTicks →
    {{Automatic, Automatic}, {ticks, None}}, FrameStyle → Directive[16],
   Joined → True, FrameLabel → {"year", "local σ (10^6 km)"},
   PlotStyle → {{Dotted, Gray}, {Dotted, Gray}},
   Filling → {1 → {2}}, PlotRange → All];
PL3 = Show[PL2, PL1, QQ3]
```




```
Show[PL2, PL1]
```



Appendix F

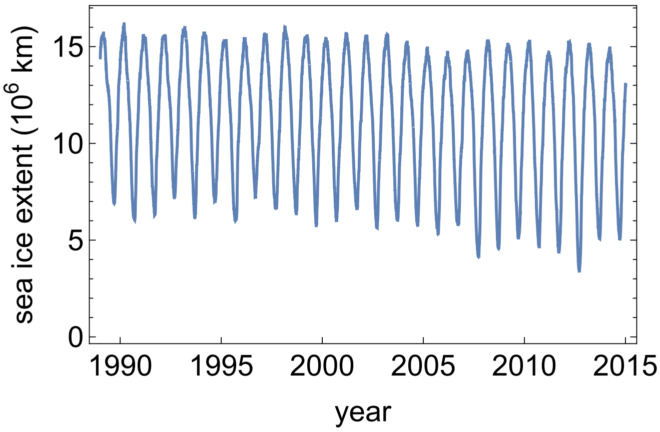
Linear model

The following code contains all the calculations and plots for real data and linear model of future predictions of sea ice extent in the Arctic, discussed in section "Linear model" in the chapter "Arctic temperature". The code was written in the program "Mathematica".

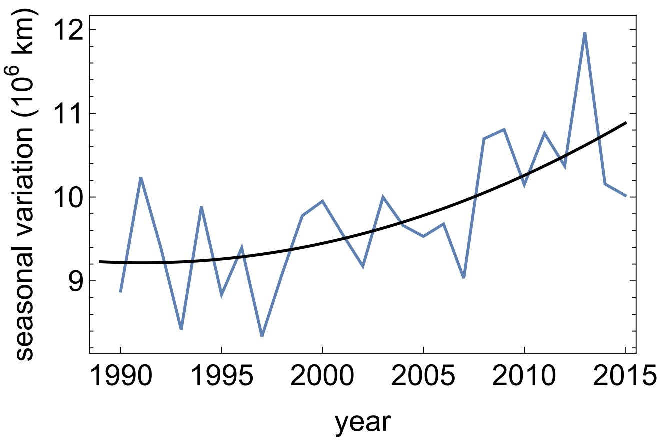
Reading and preparing data:

Read sea ice data:

```
x = ReadList[
   "Dropbox/Master Theses/sea ice/NH_seaice_extent_final.csv", String];
x = Table[ToExpression[StringSplit[x[[t]], ","][[4]]], {t, 3, Length[x]}];
ticks = {{359, "1990"}, {359 + 5 * 365, "1995"}, {359 + 10 * 365, "2000"},
   {359 + 15 * 365, "2005"}, {359 + 20 * 365, "2010"}, {359 + 25 * 365, "2015"}};
y = Drop[x, 2096];
ListPlot[y, Frame → True, FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "sea ice extent (10^6 km)"}]
```

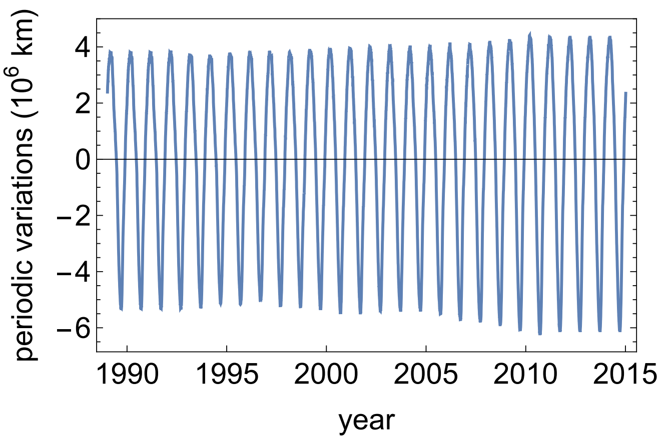



```
amplitudes = Map[Max[#] - Min[#] &, Partition[y, 365]];
fit = Fit[amplitudes, {zz^2, zz, 1}, zz];
PL1 = ListPlot[amplitudes, Joined → True];
PL2 = Plot[fit, {zz, 0, 26}, PlotStyle → Black];
newticks = {{1 + 0, "1990"}, {1 + 5, "1995"},
   {1 + 10, "2000"}, {1 + 15, "2005"}, {1 + 20, "2010"}, {1 + 25, "2015"}};
Show[{PL1, PL2}, Frame → True, FrameTicks →
  {{Automatic, Automatic}, {newticks, None}}, FrameStyle → Directive[16],
 FrameLabel → {"year", "seasonal variation (10^6 km)"}]
```



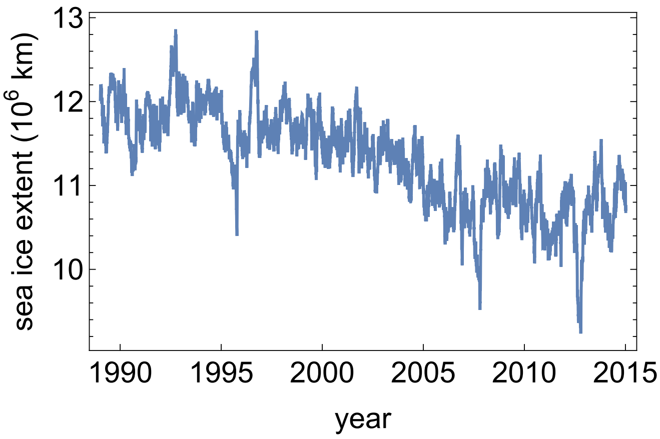
```
clim1 = Map[Mean[#] &, Transpose[Partition[y, 365][[1 ;; 5]]]];
clim2 = Map[Mean[#] &, Transpose[Partition[y, 365][[1 ;; 5]]]];
clim3 = Map[Mean[#] &, Transpose[Partition[y, 365][[1 ;; 5]]]];
clim4 = Map[Mean[#] &, Transpose[Partition[y, 365][[2 ;; 6]]]];
clim5 = Map[Mean[#] &, Transpose[Partition[y, 365][[3 ;; 7]]]];
clim6 = Map[Mean[#] &, Transpose[Partition[y, 365][[4 ;; 8]]]];
clim7 = Map[Mean[#] &, Transpose[Partition[y, 365][[5 ;; 9]]]];
clim8 = Map[Mean[#] &, Transpose[Partition[y, 365][[6 ;; 10]]]];
clim9 = Map[Mean[#] &, Transpose[Partition[y, 365][[7 ;; 11]]]];
clim10 = Map[Mean[#] &, Transpose[Partition[y, 365][[8 ;; 12]]]];
clim11 = Map[Mean[#] &, Transpose[Partition[y, 365][[9 ;; 13]]]];
clim12 = Map[Mean[#] &, Transpose[Partition[y, 365][[10 ;; 14]]]];
clim13 = Map[Mean[#] &, Transpose[Partition[y, 365][[11 ;; 15]]]];
clim14 = Map[Mean[#] &, Transpose[Partition[y, 365][[12 ;; 16]]]];
clim15 = Map[Mean[#] &, Transpose[Partition[y, 365][[13 ;; 17]]]];
clim16 = Map[Mean[#] &, Transpose[Partition[y, 365][[14 ;; 18]]]];
clim17 = Map[Mean[#] &, Transpose[Partition[y, 365][[15 ;; 19]]]];
clim18 = Map[Mean[#] &, Transpose[Partition[y, 365][[16 ;; 20]]]];
clim19 = Map[Mean[#] &, Transpose[Partition[y, 365][[17 ;; 21]]]];
clim20 = Map[Mean[#] &, Transpose[Partition[y, 365][[18 ;; 22]]]];
clim21 = Map[Mean[#] &, Transpose[Partition[y, 365][[19 ;; 23]]]];
clim22 = Map[Mean[#] &, Transpose[Partition[y, 365][[20 ;; 24]]]];
clim23 = Map[Mean[#] &, Transpose[Partition[y, 365][[21 ;; 25]]]];
clim24 = Map[Mean[#] &, Transpose[Partition[y, 365][[22 ;; 26]]]];
clim25 = Map[Mean[#] &, Transpose[Partition[y, 365][[22 ;; 26]]]];
clim26 = Map[Mean[#] &, Transpose[Partition[y, 365][[22 ;; 26]]]];
climtab = {clim1, clim2, clim3, clim4, clim5, clim6, clim7, clim8, clim9,
   clim10, clim11, clim12, clim13, clim14, clim15, clim16, clim17, clim18,
   clim19, clim20, clim21, clim22, clim23, clim24, clim25, clim26};
climtab = Map[# - Mean[#] &, climtab];
factors = Map[Max[#] - Min[#] &, climtab];
climseries = Flatten[climtab];
```

```
ListPlot[climseries, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "periodic variations (10^6 km)"}]
```



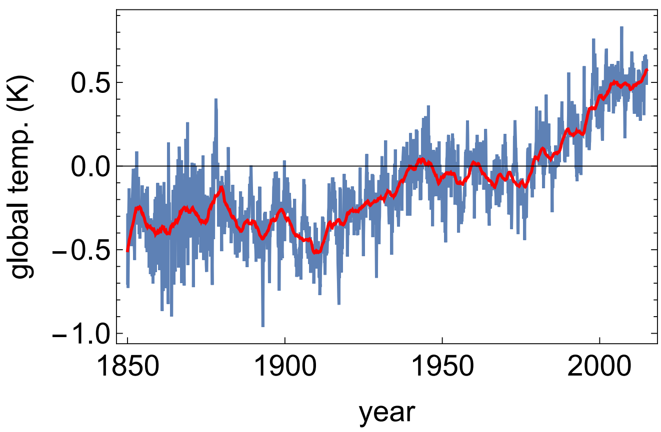
Remove climatology:

```
xx = y - climseries;
years = Partition[xx, 365];
yearmeans = Map[Mean[#] &, years];
years = Table[(years[[i]] - yearmeans[[i]]) * (factors[[i]] / Mean[factors]) +
    yearmeans[[i]], {i, 1, 26}];
xx = Flatten[years];
Q1 = ListPlot[xx, Frame → True,
  FrameTicks → {{Automatic, Automatic}, {ticks, None}},
  FrameStyle → Directive[16], Joined → True,
  FrameLabel → {"year", "sea ice extent (10^6 km)"}]
```

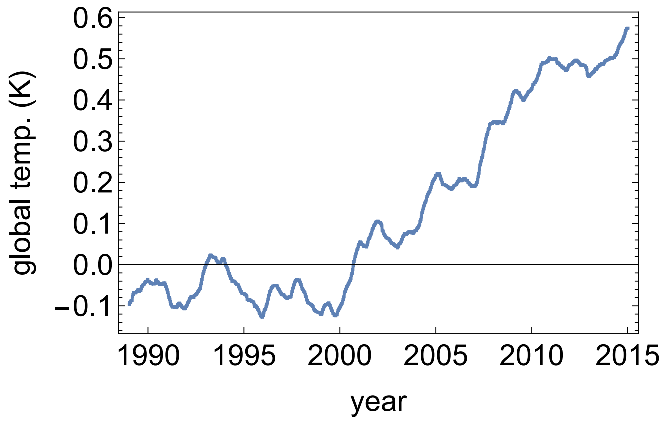


Global temperature anomaly:

```
T = ToExpression[
   StringSplit[Drop[ReadList["Dropbox/Master Theses/sea ice/globaltemp.txt",
       String], -1]][[All, 2]]];
ticks2 = {{1, "1850"}, {1 + 12 * 50, "1900"}, {1 + 12 * 100, "1950"},
   {1 + 12 * 150, "2000"}};
PL1 = ListPlot[T, Frame → True, FrameTicks →
    {{Automatic, Automatic}, {ticks2, None}}, FrameStyle → Directive[16],
   Joined → True, FrameLabel → {"year", "global temp. (K)"}];
av = Drop[MovingAverage[ArrayPad[T, 5 * 12 / 2, "Fixed"], 5 * 12], -1];
PL2 = ListPlot[av, PlotStyle → Red, Joined → True];
Show[PL1, PL2]
```



```
R = Flatten[Table[Table[av[[t]], {12}], {t, 1, Length[av]}]];
R = R[[Length[R] - Length[xx] + 1 ;; Length[R]]];
ListPlot[R, Frame → True, FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "global temp. (K)"}]
```



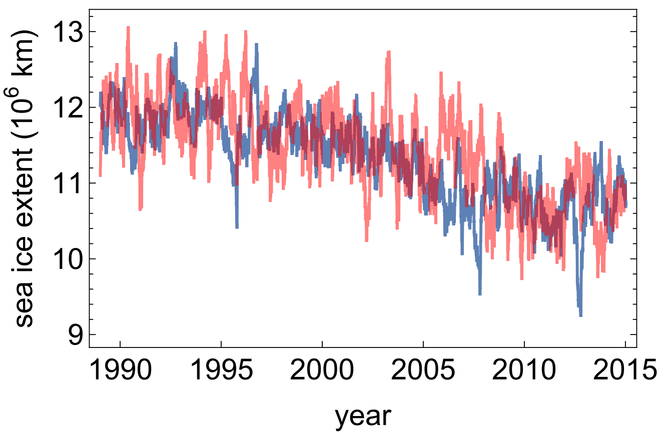
Modeling

```
OU = xx[[3000 ;; 6000]];
ΔOU = Drop[OU, 1] - Drop[OU, -1];
ΔOU = ΔOU - Mean[ΔOU];
OU = FoldList[Plus, 0, ΔOU];
OU = OU - Mean[OU];
OU = Thread[{#1, #2} &[Range[Length[OU]], OU]];
est = EstimatedProcess[OU, OrnsteinUhlenbeckProcess[μ, Σ, θ]]
```

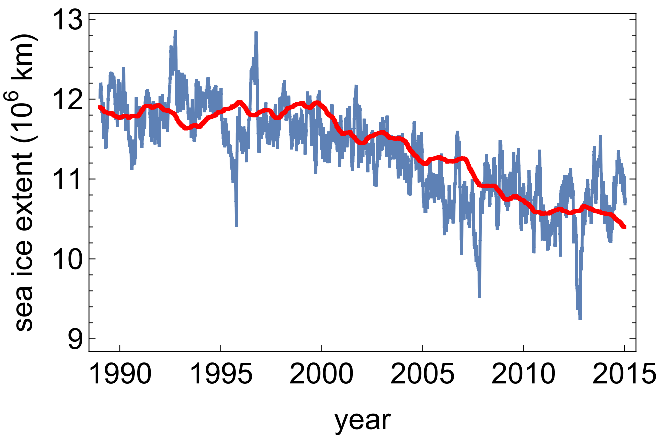
```
OrnsteinUhlenbeckProcess[0.0000320507, 0.26038, 0.0182579]
```

```
a = 0.018;
Clear[t];
σ = 0.017;
r0 = 0.92;
v = 0.15;
A = 21.1;
B = 10;
r = r0;
tab = {1};
rliste = {r};
pliste = {};
driver = (R - First[R]) / (Last[R] - First[R]);
Monitor[
 Do[
   rand = RandomReal[NormalDistribution[0, σ], 5 * 31];
   rand = Thread[{#1, #2} & [Range[5 * 31] / 5., rand]];
   ifun  = Interpolation[rand];
   (* xxxxxxxx *)
   r = r0 + v * driver[[30 * t]];
   rliste = Append[rliste, r];
   Q0 = Last[tab];
   s = NDSolve[{Q'[tt] == a * (r - Q[tt]) + ifun[tt + 1], Q[0] == Q0}, Q, {tt, 0, 30}];
   mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
   tab = Join[tab, mid];
   , {t, 1, 316}];
 , t]
icemodel = -tab;
```

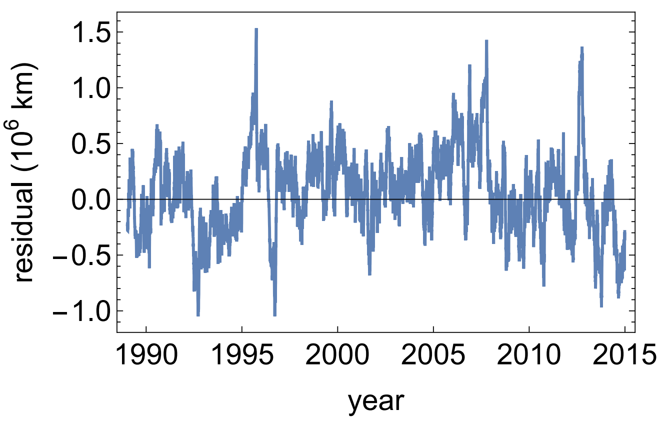
```
Q2 = ListPlot[A + B * (icemodel), Joined → True,
   PlotRange → All, PlotStyle → {Red, Opacity[0.5]}];
modelexample = A + B * (icemodel);
Show[{Q1, Q2}, PlotRange → All]
```



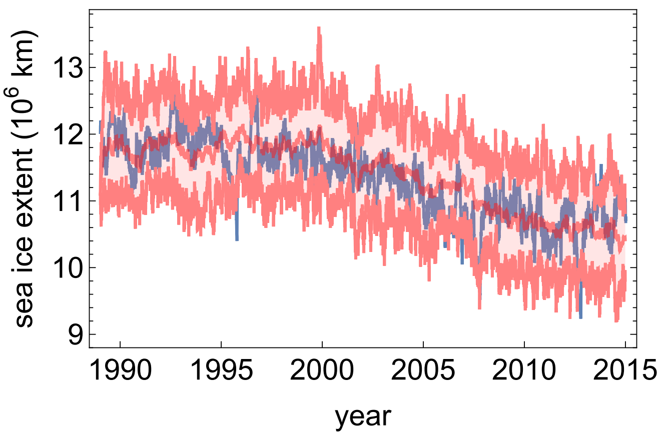
```
Clear[r]
r0 = 0.92;
v = 0.15;
A = 11.1;
B = 10;
fix[r_] := (r - 1);
Q3 = ListPlot[A - B * Map[fix[#] &, r0 + v * driver[[Range[30 * 316]]]],
   PlotStyle → {Red, Thick}, Joined → True];
Show[{Q1, Q3}, PlotRange -> All]
```




```
residual = (A - B * Map[fix[#] &, r0 + v * driver[[Range[30 * 316]]]]) - Drop[xx, 10];
ListPlot[residual, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "residual (10^6 km)"}]
```



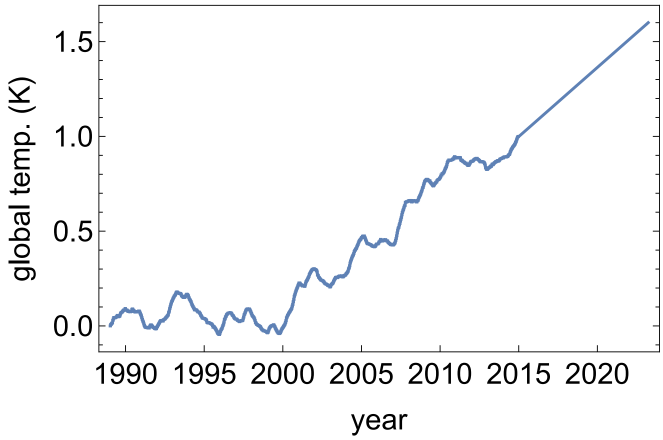
```
mlist = {};
Monitor[
  Do[
    Clear[t];
    r = r0;
    tab = {1};
    rliste = {r};
    pliste = {};
    driver = (R - First[R]) / (Last[R] - First[R]);
    Monitor[
     Do[
       rand = RandomReal[NormalDistribution[0, σ], 5 * 31];
       rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
       ifun  = Interpolation[rand];
       (* xxxxxxxx *)
       r = r0 + v * driver[[30 * t]];
       rliste = Append[rliste, r];
       Q0 = Last[tab];
       s = NDSolve[
         {Q'[tt] == a * (r - Q[tt]) + ifun[tt + 1], Q[0] == Q0}, Q, {tt, 0, 30}];
       mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
       tab = Join[tab, mid];
       , {t, 1, 316}];
     , t];
    icemodel = -tab;
    model = A + B * (icemodel);
    mlist = Append[mlist, model];
    , {j, 1, 20}];
  , j];

mean = Map[Mean[#] &, Transpose[mlist]];
low = Map[Quantile[#, 0.025] &, Transpose[mlist]];
high = Map[Quantile[#, 1 - 0.025] &, Transpose[mlist]];


Qmean = ListPlot[mean, Joined → True,
    PlotRange → All, PlotStyle → {Red, Opacity[0.5]}];
Qampl = ListPlot[{low, high}, Joined → True, PlotRange → All,
    PlotStyle → {Pink}, Filling → {1 → {2}}];
Q11 = ListPlot[xx, Frame → True, FrameTicks →
     {{Automatic, Automatic}, {ticks, None}}, FrameStyle → Directive[16],
    Joined → True, FrameLabel → {"year", "sea ice extent (10^6 km)"}];
Show[{Q11, Qmean, Qampl}, PlotRange → All]
```



Future prediction

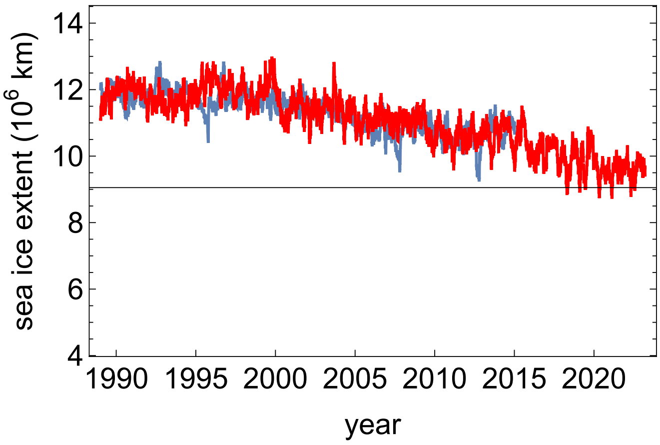
Scenario for temperature:

```
ticks = {{359, "1990"}, {359 + 5 * 365, "1995"},
    {359 + 10 * 365, "2000"}, {359 + 15 * 365, "2005"}, {359 + 20 * 365, "2010"},
    {359 + 25 * 365, "2015"}, {359 + 30 * 365, "2020"}};
newdriver = Join[driver, Last[driver] + 0.0002 * Range[3000]];
ListPlot[newdriver, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "global temp. (K)"}]
```




```
Clear[t];
r = r0;
tab = {1};
rliste = {r};
pliste = {};
driver = (R - First[R]) / (Last[R] - First[R]);
Monitor[
 Do[
   rand = RandomReal[NormalDistribution[0, σ], 5 * 31];
   rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
   ifun = Interpolation[rand];
   (* xxxxxxxx *)
   r = r0 + v * newdriver[[30 * t]];
   rliste = Append[rliste, r];
   Q0 = Last[tab];
   s = NDSolve[{Q'[tt] == a * (r - Q[tt]) + ifun[tt + 1], Q[0] == Q0}, Q, {tt, 0, 30}];
   mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
   tab = Join[tab, mid];
   , {t, 1, 316 + 100}];
 , t]
icemodel = -tab;
Q3 =
  ListPlot[A + B * (icemodel), Joined → True, PlotRange → All, PlotStyle → Red];
```

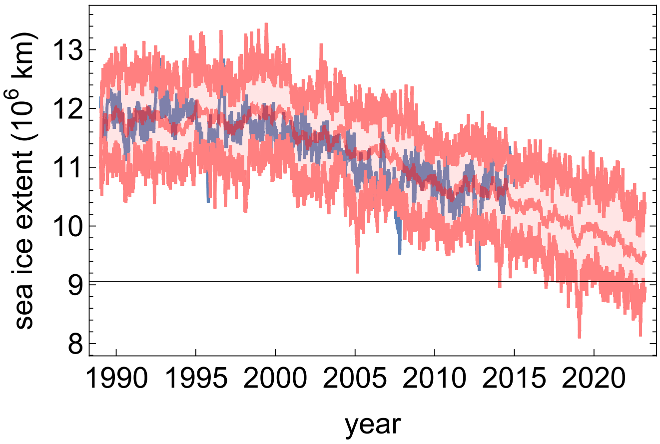
```
Show[{Q1, Q3}, PlotRange → {4.5, 14},
 FrameTicks → {{Automatic, Automatic}, {ticks, None}}]
```




```
mlist = {};
Monitor[
  Do[
    Clear[t];
    r = r0;
    tab = {1};
    rliste = {r};
    pliste = {};
    driver = (R - First[R]) / (Last[R] - First[R]);
    Monitor[
     Do[
       rand = RandomReal[NormalDistribution[0, σ], 5 * 31];
       rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
       ifun  = Interpolation[rand];
       (* xxxxxxxx *)
       r = r0 + v * newdriver[[30 * t]];
       rliste = Append[rliste, r];
       Q0 = Last[tab];
       s = NDSolve[
          {Q'[tt] == a * (r - Q[tt]) + ifun[tt + 1], Q[0] == Q0}, Q, {tt, 0, 30}];
       mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
       tab = Join[tab, mid];
       , {t, 1, 316 + 100}];
     , t];
    icemodel = -tab;
    model = A + B * (icemodel);
    mlist = Append[mlist, model];
    , {j, 1, 20}];
  , j];
```

```
mean = Map[Mean[#] &, Transpose[mlist]];
low = Map[Quantile[#, 0.025] &, Transpose[mlist]];
high = Map[Quantile[#, 1 - 0.025] &, Transpose[mlist]];
```

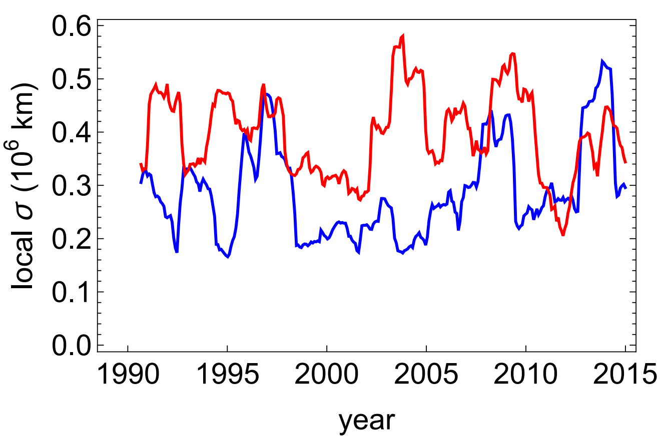
```
Qmean = ListPlot[mean, Joined → True,
   PlotRange → All, PlotStyle → {Red, Opacity[0.5]}];
Qampl = ListPlot[{low, high}, Joined → True, PlotRange → All,
   PlotStyle → {Pink}, Filling → {1 → {2}}];
Q11 = ListPlot[xx, Frame → True, FrameTicks →
     {{Automatic, Automatic}, {ticks, None}}, FrameStyle → Directive[16],
   Joined → True, FrameLabel → {"year", "sea ice extent (10^6 km)"}];
Show[{Q11, Qmean, Qampl}, PlotRange → All]
```



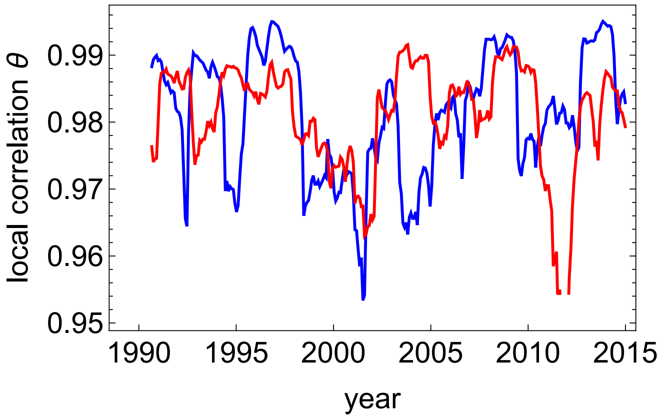
Early warning indications

Estimate variance and correlation. Red for model. Blue for real data.

```
win = 600;
σlist = {};
τlist = {};
Monitor[
  Do[
    qqq = modelexample[[t - win ;; t]];
    σloc = StandardDeviation[qqq]; (*local standard deviation*)
    τloc = Correlation[Drop[qqq, 1], Drop[qqq, -1]];
    (*local correlation scale*)
    σlist = Append[σlist, {t, σloc}];
    τlist = Append[τlist, {t, τloc}];
    , {t, win + 1, Length[modelexample], 30}];
  , t];
QQ1 = ListPlot[σlist, Frame → True,
   FrameTicks → {{Automatic, Automatic}, {ticks, None}},
   FrameStyle → Directive[16], Joined → True,
   FrameLabel → {"year", "local σ (10^6 km)"}, PlotStyle → Red];
QQ2 = ListPlot[τlist, Frame → True, FrameTicks →
    {{Automatic, Automatic}, {ticks, None}}, FrameStyle → Directive[16],
   Joined → True, FrameLabel → {"year", "local correlation θ"}, PlotStyle → Red];
win = 600;
σlist = {};
τlist = {};
Monitor[
  Do[
    qqq = xx[[t - win ;; t]];
    σloc = StandardDeviation[qqq]; (*local standard deviation*)
    τloc = Correlation[Drop[qqq, 1], Drop[qqq, -1]];
    (*local correlation scale*)
    σlist = Append[σlist, {t, σloc}];
    τlist = Append[τlist, {t, τloc}];
    , {t, win + 1, Length[modelexample], 30}];
  , t];
QQ3 = ListPlot[σlist, Frame → True,
   FrameTicks → {{Automatic, Automatic}, {ticks, None}},
   FrameStyle → Directive[16], Joined → True,
   FrameLabel → {"year", "local σ (10^6 km)"}, PlotStyle → Blue];
QQ4 = ListPlot[τlist, Frame → True,
   FrameTicks → {{Automatic, Automatic}, {ticks, None}},
   FrameStyle → Directive[16], Joined → True,
   FrameLabel → {"year", "local correlation θ"}, PlotStyle → Blue];
Show[{QQ3, QQ1}, PlotRange → All]
```



```
Show[{QQ4, QQ2}, PlotRange → All]
```

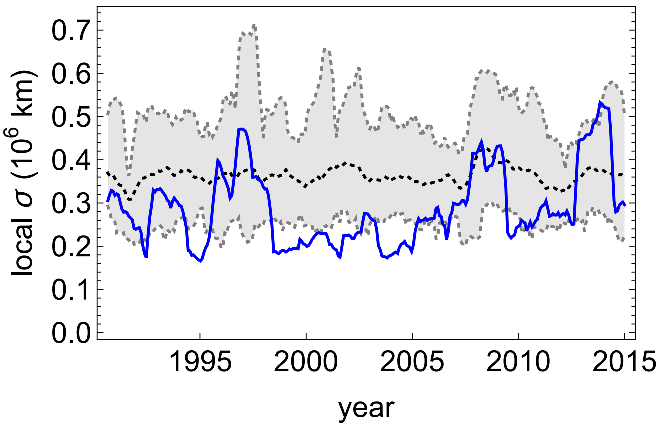


Monte Carlo

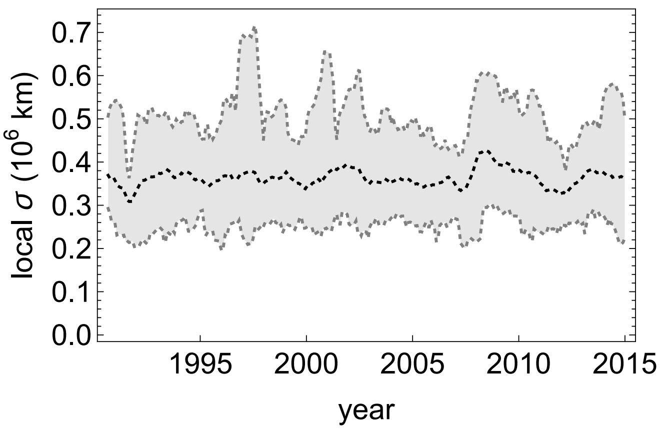
```
σlistlist = {};
τlistlist = {};
```

```
Monitor[
 Do[
   Clear[t];
   r = r0;
   tab = {1};
   rliste = {r};
   pliste = {};
   driver = (R - First[R]) / (Last[R] - First[R]);
   Monitor[
     Do[
       rand = RandomReal[NormalDistribution[0, σ], 5 * 31];
       rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
       ifun = Interpolation[rand];
       (* xxxxxxxx *)
       r = r0 + v * driver[[30 * t]];
       rliste = Append[rliste, r];
       Q0 = Last[tab];
       s = NDSolve[
         {Q'[tt] == a * (r - Q[tt]) + ifun[tt + 1], Q[0] == Q0}, Q, {tt, 0, 30}];
       mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
       tab = Join[tab, mid];
       , {t, 1, 316}];
     , t]
    icemodel = -tab;
   modelexample = A + B * (icemodel);
   win = 600;
   σlist = {};
   τlist = {};
   Do[
    qqq = modelexample[[t - win ;; t]];
    σloc = StandardDeviation[qqq]; (*local standard deviation*)
    τloc = Correlation[Drop[qqq, 1], Drop[qqq, -1]];
    (*local correlation scale*)
    σlist = Append[σlist, {t, σloc}];
    τlist = Append[τlist, {t, τloc}];
    , {t, win + 1, Length[modelexample], 30}];
   σlistlist = Append[σlistlist, σlist];
   τlistlist = Append[τlistlist, τlist];
   , {run, 1, 20}];
 , {run, t}]
```

```
times = σlistlist[[1]][[All, 1]];
ticks = {{359, "1990"}, {359 + 5 * 365, "1995"},
   {359 + 10 * 365, "2000"}, {359 + 15 * 365, "2005"}, {359 + 20 * 365, "2010"},
   {359 + 25 * 365, "2015"}, {359 + 30 * 365, "2020"}};
σmean = Map[Mean[#] &, Transpose[Map[#[[All, 2]] &, σlistlist]]];
σlow = Map[Quantile[#, 0.025] &, Transpose[Map[#[[All, 2]] &, σlistlist]]];
σhigh = Map[Quantile[#, 1 - 0.025] &, Transpose[Map[#[[All, 2]] &, σlistlist]]];

σmean = Thread[{#1, #2} &[times, σmean]];
σlow = Thread[{#1, #2} &[times, σlow]];
σhigh = Thread[{#1, #2} &[times, σhigh]];

PL1 = ListPlot[σmean, Frame → True,
   FrameTicks → {{Automatic, Automatic}, {ticks, None}},
   FrameStyle → Directive[16], Joined → True,
   FrameLabel → {"year", "local correlation θ"}, PlotStyle → {Black, Dotted}];
PL2 = ListPlot[{σlow, σhigh}, Frame → True, FrameTicks →
    {{Automatic, Automatic}, {ticks, None}}, FrameStyle → Directive[16],
   Joined → True, FrameLabel → {"year", "local σ (10^6 km)"},
   PlotStyle → {{Dotted, Gray}, {Dotted, Gray}},
   Filling → {1 → {2}}, PlotRange → All];
PL3 = Show[PL2, PL1, QQ3]
```




```
Show[PL2, PL1]
```



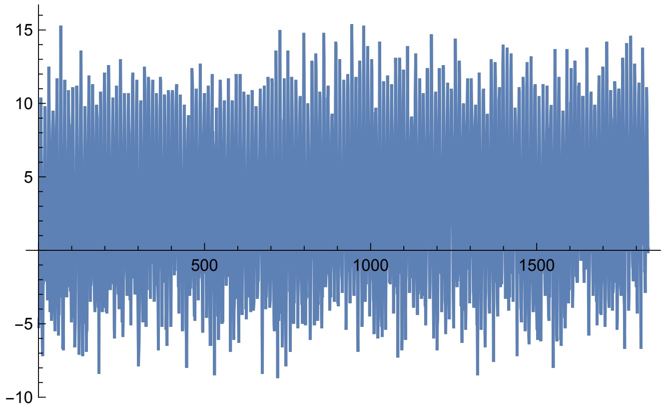
Appendix G

Mean Arctic temperature

The following code contains all the calculations and plots for mean Arctic temperatures in Tromsø, Alta, Amderma, Arkhangelsk, Greenland, Pinega, Svalbard, Tiksi and Vardø discussed in the beginning of the chapter "Arctic temperature". The code was written in the program "Mathematica".

```
T = ToExpression[Flatten[Map[Drop[#, 1] &, Map[StringSplit[#] &, ReadList[
        "Dropbox/Master Theses/ArcticData/globaltemp2.txt", String]]]]];
```

```
ListPlot[T, Joined → True]
```

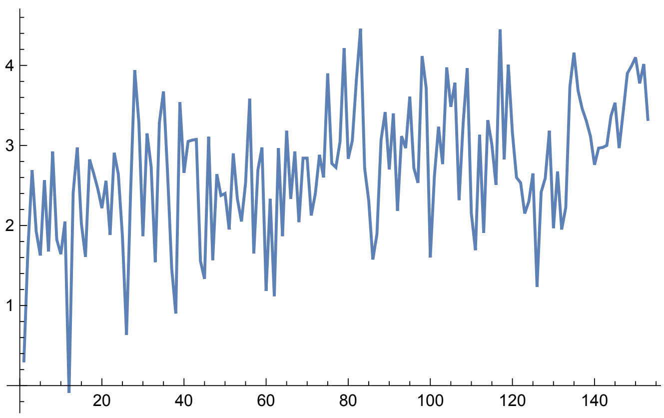



```
Export["Dropbox/Master Theses/ArcticData/tromsotemp.txt", T]
```

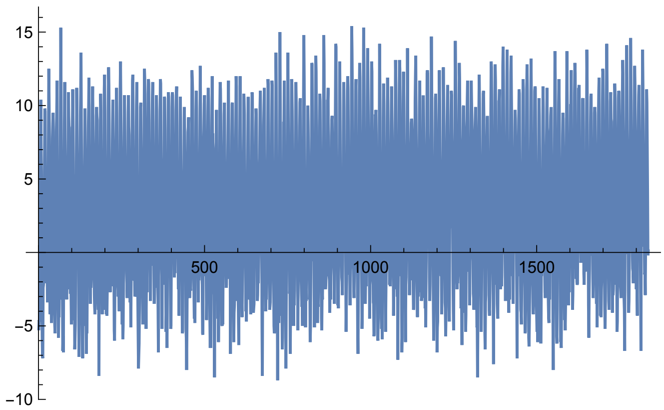
```
Dropbox/Master Theses/ArcticData/tromsotemp.txt
```

```
Map[Mean[#] &, Partition[T, 12]];
```

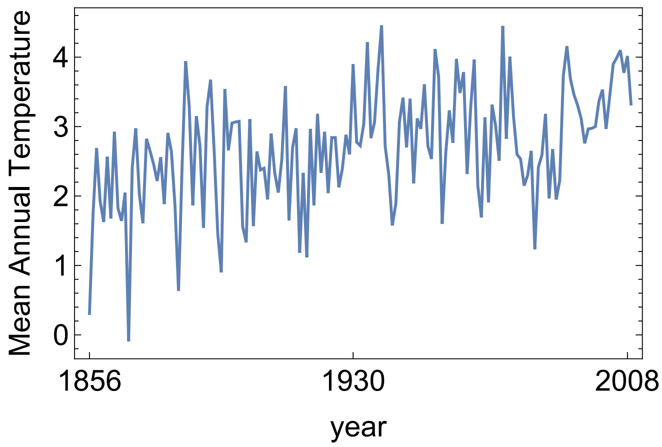
```
ListPlot[%, Joined → True]
```



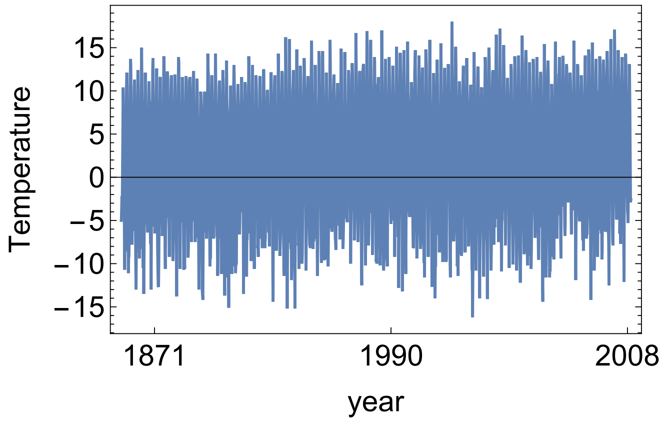
```
T1 = ToExpression[Flatten[Map[Drop[#, 1] &, Map[StringSplit[#] &,
      ReadList["Desktop/ArcticData/Tromso.txt", String]]]]];
Export["Desktop/ArcticData/tromsotemp.txt", T1];
ListPlot[T1, Joined → True]
```



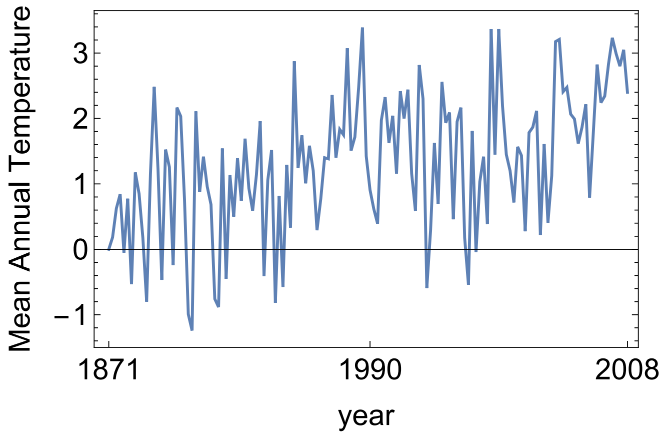

```
Meantemp1 = Map[Mean[#] &, Partition[T1, 12]];
ticks8 = {{1, "1856"}, {1 + 74, "1930"}, {1 + 151, "2008"}};
ListPlot[Meantemp1, Joined → True, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks8, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "Mean Annual Temperature"}]
```



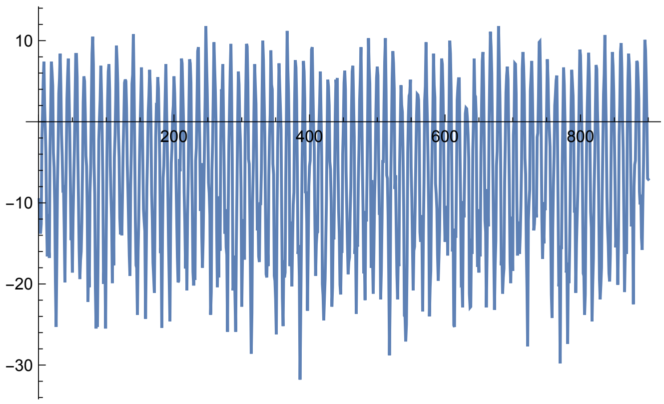
```
T2 = ToExpression[Flatten[Map[Drop[#, 1] &,
     Map[StringSplit[#] &, ReadList["Desktop/ArcticData/Alta.txt", String]]]]];
Export["Desktop/ArcticData/Altatemp.txt", T2];
ticks = {{109, "1871"}, {109 + 64 * 12, "1990"}, {109 + 128 * 12, "2008"}};
ListPlot[T2, Joined → True, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True, FrameLabel → {"year", "Temperature"}]
```



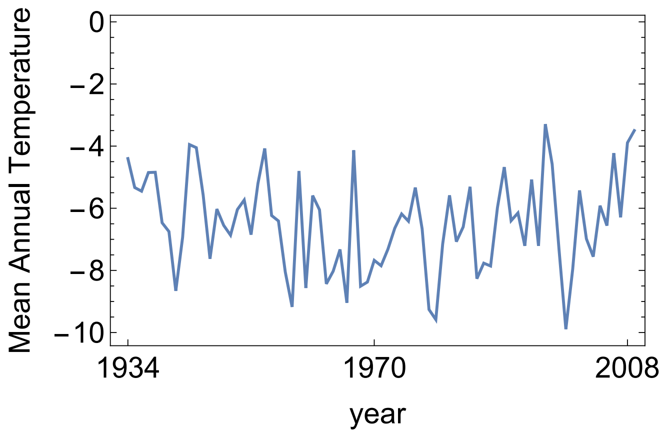

```
Meantemp2 = Map[Mean[#] &, Partition[T2, 12]];
ticks1 = {{1, "1871"}, {1 + 69, "1990"}, {1 + 137, "2008"}};
ListPlot[Meantemp2, Joined → True, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks1, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "Mean Annual Temperature"}]
```



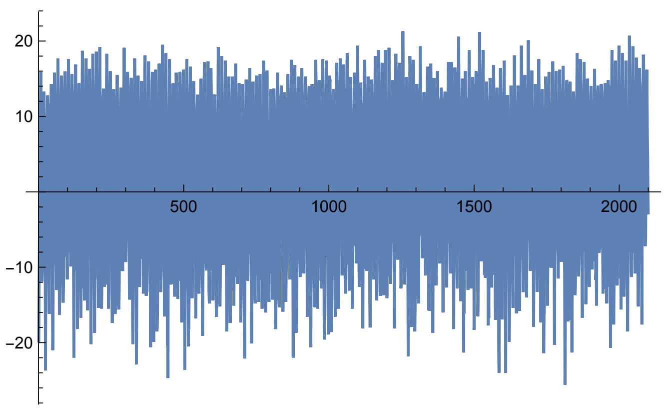
```
T3 = ToExpression[Flatten[Map[Drop[#, 1] &, Map[StringSplit[#] &,
      ReadList["Desktop/ArcticData/Amderma.txt", String]]]]];
Export["Desktop/ArcticData/Amdermatemp.txt", T3];
ListPlot[T3, Joined → True]
```



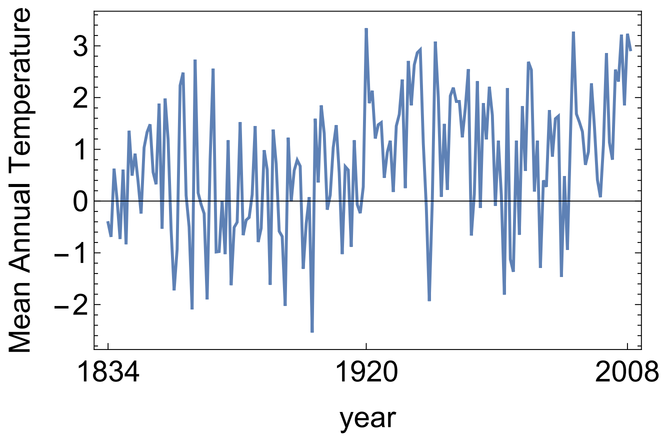

```
Meantemp3 = Map[Mean[#] &, Partition[T3, 12]];
ticks2 = {{1, "1934"}, {1 + 36, "1970"}, {1 + 73, "2008"}};
ListPlot[Meantemp3, Joined → True, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks2, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "Mean Annual Temperature"}]
```



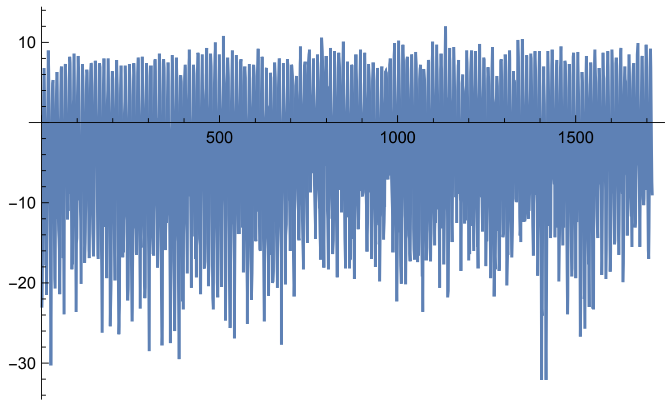

```
T4 = ToExpression[Flatten[Map[Drop[#, 1] &, Map[StringSplit[#] &,
      ReadList["Desktop/ArcticData/Arkhangelsk.txt", String]]]]];
Export["Desktop/ArcticData/Arkhangelsktemp.txt", T4];
ListPlot[T4, Joined → True]
```



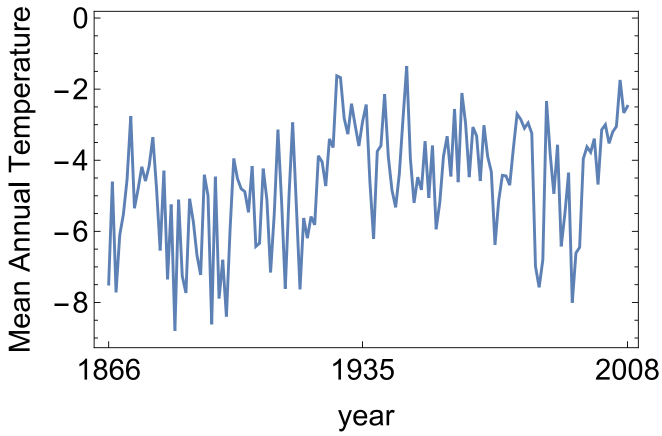
```
Meantemp4 = Map[Mean[#] &, Partition[T4, 12]];
ticks3 = {{1, "1834"}, {1 + 86, "1920"}, {1 + 173, "2008"}};
ListPlot[Meantemp4, Joined → True, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks3, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "Mean Annual Temperature"}]
```



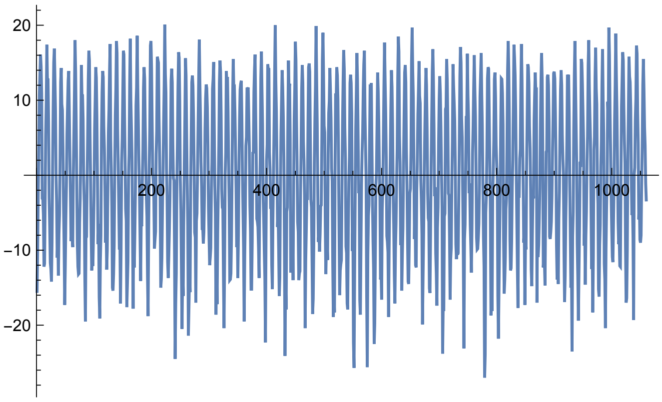

```
T5 = ToExpression[Flatten[Map[Drop[#, 1] &, Map[StringSplit[#] &,
      ReadList["Desktop/ArcticData/Greenland.txt", String]]]]];
Export["Desktop/ArcticData/Greenlandtemp.txt", T5];
ListPlot[T5, Joined → True]
```



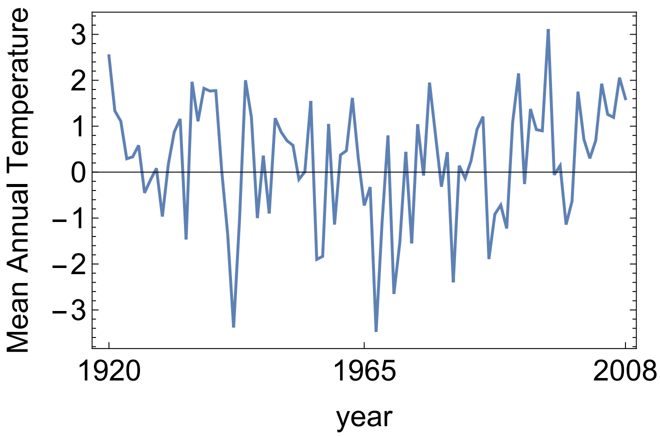
```
Meantemp5 = Map[Mean[#] &, Partition[T5, 12]];
ticks4 = {{1, "1866"}, {1 + 69, "1935"}, {1 + 141, "2008"}};
ListPlot[Meantemp5, Joined → True, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks4, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "Mean Annual Temperature"}]
```



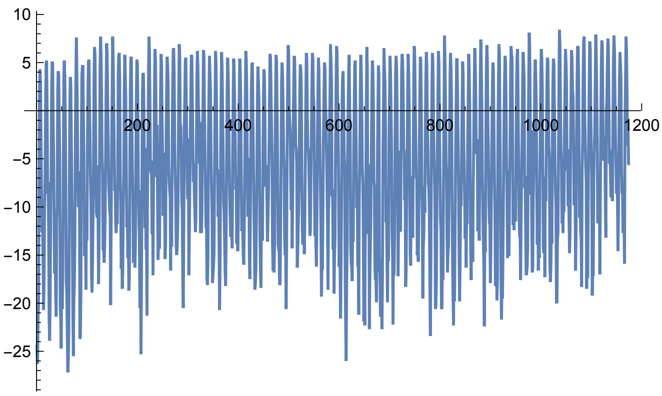

```
T6 = ToExpression[Flatten[Map[Drop[#, 1] &, Map[StringSplit[#] &,
      ReadList["Desktop/ArcticData/Pinega.txt", String]]]]];
Export["Desktop/ArcticData/Pinegatemp.txt", T6];
ListPlot[T6, Joined → True]
```



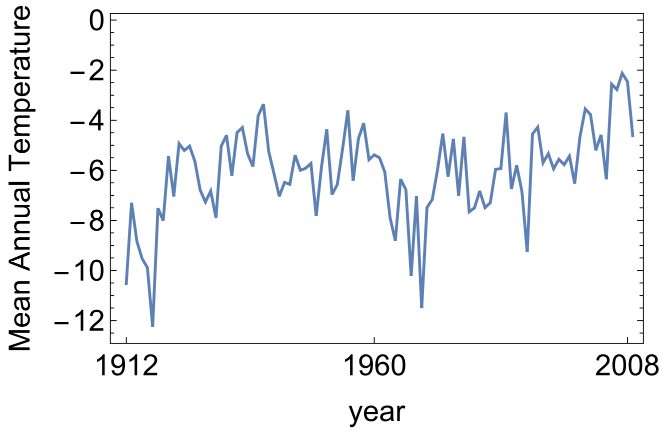
```
Meantemp6 = Map[Mean[#] &, Partition[T6, 12]];
ticks5 = {{1, "1920"}, {1 + 43, "1965"}, {1 + 87, "2008"}};
ListPlot[Meantemp6, Joined → True, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks5, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "Mean Annual Temperature"}]
```



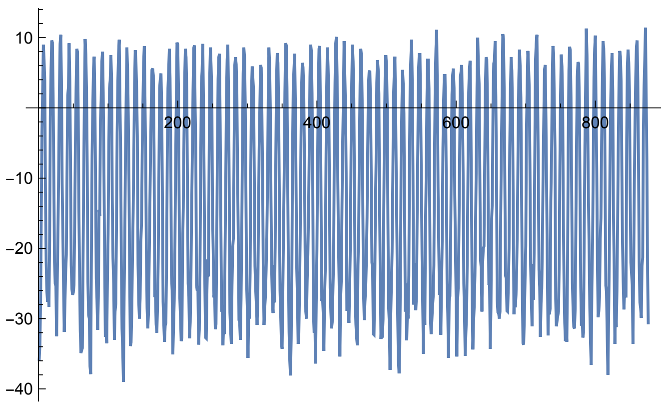

```
T8 = ToExpression[Flatten[Map[Drop[#, 1] &, Map[StringSplit[#] &,
      ReadList["Desktop/ArcticData/Svalbard.txt", String]]]]];
Export["Desktop/ArcticData/Svalbardtemp.txt", T8];
ListPlot[T8, Joined → True]
```



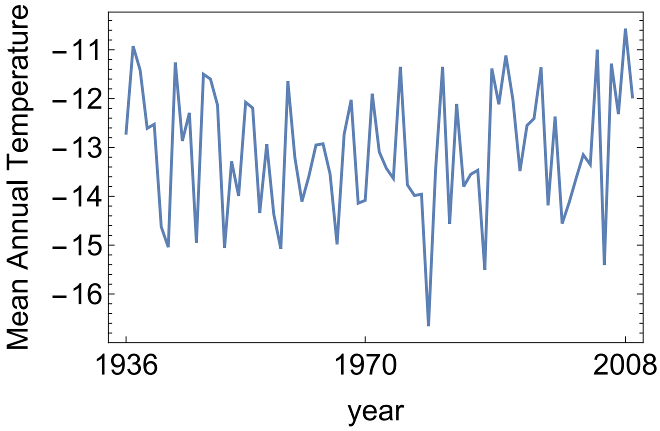
```
Meantemp8 = Map[Mean[#] &, Partition[T8, 12]];
ticks6 = {{1, "1912"}, {1 + 47, "1960"}, {1 + 95, "2008"}};
ListPlot[Meantemp8, Joined → True, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks6, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "Mean Annual Temperature"}]
```



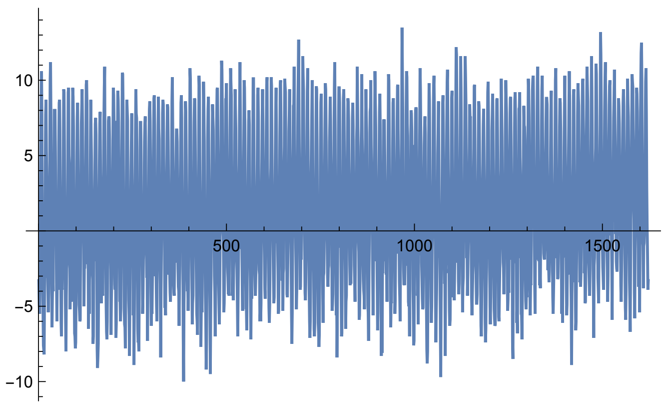

```
T9 = ToExpression[Flatten[Map[Drop[#, 1] &, Map[StringSplit[#] &,
      ReadList["Desktop/ArcticData/Tiksi.txt", String]]]]];
Export["Desktop/ArcticData/Tiksitemp.txt", T9];
ListPlot[T9, Joined → True]
```



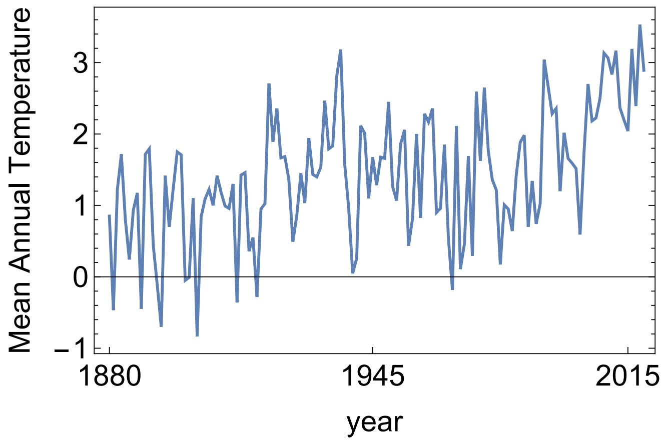
```
Meantemp9 = Map[Mean[#] &, Partition[T9, 12]];
ticks7 = {{1, "1936"}, {1 + 34, "1970"}, {1 + 71, "2008"}};
ListPlot[Meantemp9, Joined → True, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks7, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "Mean Annual Temperature"}]
```




```
T10 = ToExpression[Flatten[Map[Drop[#, 1] &, Map[StringSplit[#] &,
      ReadList["Desktop/ArcticData/Vardo.txt", String]]]]];
Export["Desktop/ArcticData/Vardotemp.txt", T10];
ListPlot[T10, Joined → True]
```



```
Meantemp10 = Map[Mean[#] &, Partition[T10, 12]];
ticks10 = {{1, "1880"}, {1 + 66, "1945"}, {1 + 130, "2015"}};
ListPlot[Meantemp10, Joined → True, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks10, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "Mean Annual Temperature"}]
```



Appendix H

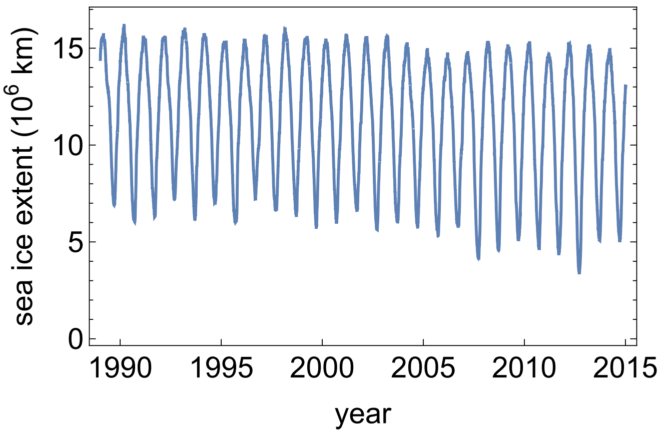
Vardø temperature and white noise

The following code contains all the calculations and plots for real data and modeling of future predictions of sea ice extent in the Arctic using temperature in Vardø as a driver and white noise as an additive noise, discussed in section "White noise" in the chapter "Arctic temperature". The code was written in the program "Mathematica".

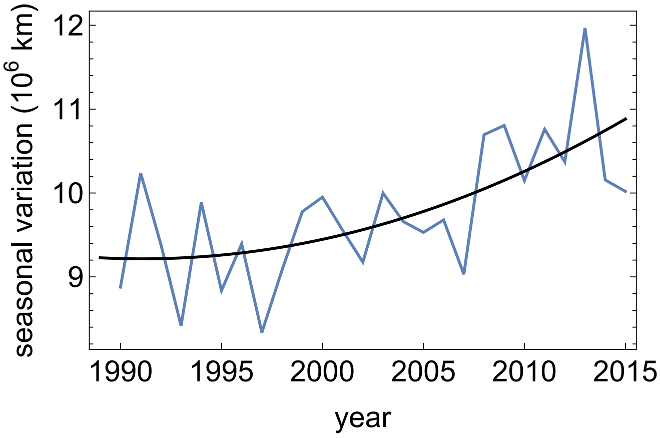
Reading and preparing data:

Read sea ice data:

```
x = ReadList[
   "Dropbox/Master Theses/sea ice/NH_seaice_extent_final.csv", String];
x = Table[ToExpression[StringSplit[x[[t]], ","][[4]]], {t, 3, Length[x]}];
ticks = {{359, "1990"}, {359 + 5 * 365, "1995"}, {359 + 10 * 365, "2000"},
   {359 + 15 * 365, "2005"}, {359 + 20 * 365, "2010"}, {359 + 25 * 365, "2015"}};
y = Drop[x, 2096];
ListPlot[y, Frame → True, FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "sea ice extent (10^6 km)"}]
```

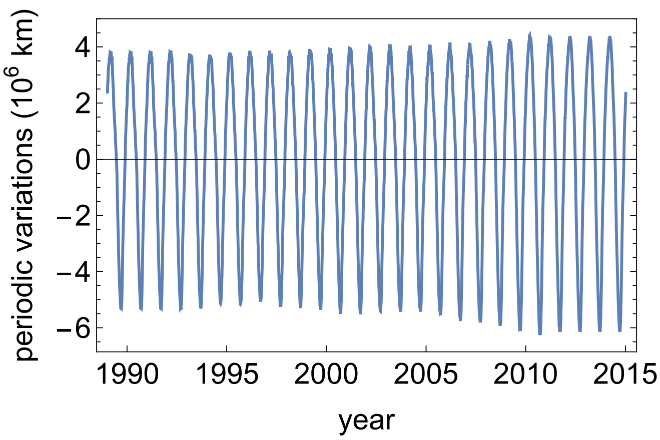



```
amplitudes = Map[Max[#] - Min[#] &, Partition[y, 365]];
fit = Fit[amplitudes, {zz^2, zz, 1}, zz];
PL1 = ListPlot[amplitudes, Joined → True];
PL2 = Plot[fit, {zz, 0, 26}, PlotStyle → Black];
newticks = {{1 + 0, "1990"}, {1 + 5, "1995"},
   {1 + 10, "2000"}, {1 + 15, "2005"}, {1 + 20, "2010"}, {1 + 25, "2015"}};
Show[{PL1, PL2}, Frame → True, FrameTicks →
  {{Automatic, Automatic}, {newticks, None}}, FrameStyle → Directive[16],
 FrameLabel → {"year", "seasonal variation (10^6 km)"}]
```



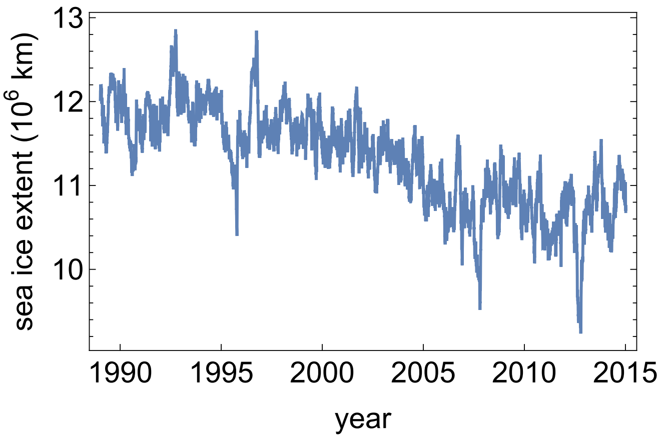
```
clim1 = Map[Mean[#] &, Transpose[Partition[y, 365][[1 ;; 5]]]];
clim2 = Map[Mean[#] &, Transpose[Partition[y, 365][[1 ;; 5]]]];
clim3 = Map[Mean[#] &, Transpose[Partition[y, 365][[1 ;; 5]]]];
clim4 = Map[Mean[#] &, Transpose[Partition[y, 365][[2 ;; 6]]]];
clim5 = Map[Mean[#] &, Transpose[Partition[y, 365][[3 ;; 7]]]];
clim6 = Map[Mean[#] &, Transpose[Partition[y, 365][[4 ;; 8]]]];
clim7 = Map[Mean[#] &, Transpose[Partition[y, 365][[5 ;; 9]]]];
clim8 = Map[Mean[#] &, Transpose[Partition[y, 365][[6 ;; 10]]]];
clim9 = Map[Mean[#] &, Transpose[Partition[y, 365][[7 ;; 11]]]];
clim10 = Map[Mean[#] &, Transpose[Partition[y, 365][[8 ;; 12]]]];
clim11 = Map[Mean[#] &, Transpose[Partition[y, 365][[9 ;; 13]]]];
clim12 = Map[Mean[#] &, Transpose[Partition[y, 365][[10 ;; 14]]]];
clim13 = Map[Mean[#] &, Transpose[Partition[y, 365][[11 ;; 15]]]];
clim14 = Map[Mean[#] &, Transpose[Partition[y, 365][[12 ;; 16]]]];
clim15 = Map[Mean[#] &, Transpose[Partition[y, 365][[13 ;; 17]]]];
clim16 = Map[Mean[#] &, Transpose[Partition[y, 365][[14 ;; 18]]]];
clim17 = Map[Mean[#] &, Transpose[Partition[y, 365][[15 ;; 19]]]];
clim18 = Map[Mean[#] &, Transpose[Partition[y, 365][[16 ;; 20]]]];
clim19 = Map[Mean[#] &, Transpose[Partition[y, 365][[17 ;; 21]]]];
clim20 = Map[Mean[#] &, Transpose[Partition[y, 365][[18 ;; 22]]]];
clim21 = Map[Mean[#] &, Transpose[Partition[y, 365][[19 ;; 23]]]];
clim22 = Map[Mean[#] &, Transpose[Partition[y, 365][[20 ;; 24]]]];
clim23 = Map[Mean[#] &, Transpose[Partition[y, 365][[21 ;; 25]]]];
clim24 = Map[Mean[#] &, Transpose[Partition[y, 365][[22 ;; 26]]]];
clim25 = Map[Mean[#] &, Transpose[Partition[y, 365][[22 ;; 26]]]];
clim26 = Map[Mean[#] &, Transpose[Partition[y, 365][[22 ;; 26]]]];
climtab = {clim1, clim2, clim3, clim4, clim5, clim6, clim7, clim8, clim9,
   clim10, clim11, clim12, clim13, clim14, clim15, clim16, clim17, clim18,
   clim19, clim20, clim21, clim22, clim23, clim24, clim25, clim26};
climtab = Map[# - Mean[#] &, climtab];
factors = Map[Max[#] - Min[#] &, climtab];
climseries = Flatten[climtab];
```

```
ListPlot[climseries, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "periodic variations (10^6 km)"}]
```



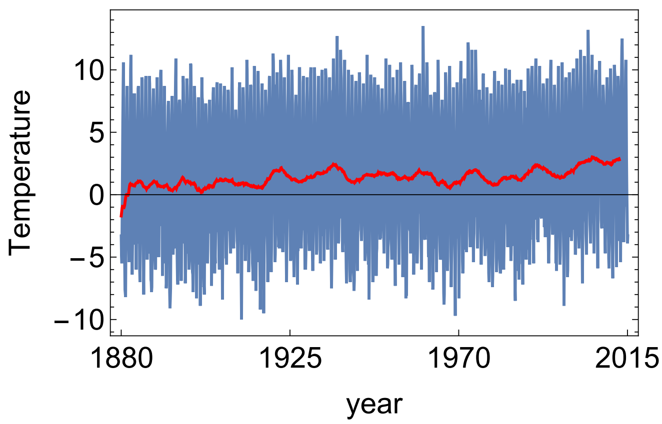
Remove climatology:

```
xx = y - climseries;
years = Partition[xx, 365];
yearmeans = Map[Mean[#] &, years];
years = Table[(years[[i]] - yearmeans[[i]]) * (factors[[i]] / Mean[factors]) +
    yearmeans[[i]], {i, 1, 26}];
xx = Flatten[years];
Q1 = ListPlot[xx, Frame → True,
  FrameTicks → {{Automatic, Automatic}, {ticks, None}},
  FrameStyle → Directive[16], Joined → True,
  FrameLabel → {"year", "sea ice extent (10^6 km)"}]
```

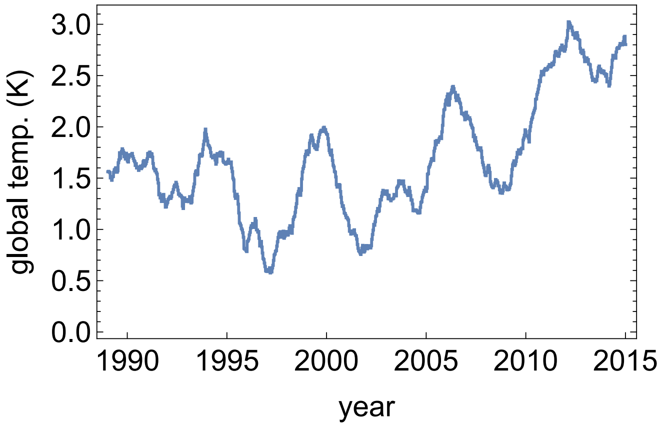


Vardo temperature anomaly:

```
T = ReadList["Dropbox/Master Theses/Vardo_temperature/Vardotemp.txt"];
ticks2 =
  {{1, "1880"}, {1 + 12 * 45, "1925"}, {1 + 12 * 90, "1970"}, {1 + 12 * 135, "2015"}};
PL1 = ListPlot[T, Frame → True, FrameTicks →
    {{Automatic, Automatic}, {ticks2, None}}, FrameStyle → Directive[16],
   Joined → True, FrameLabel → {"year", "Temperature"}];
av = Drop[MovingAverage[ArrayPad[T, 5 * 12 / 2, "Fixed"], 5 * 12], -28];
PL2 = ListPlot[av, PlotStyle → Red, Joined → True];
Length[T]
Show[PL1, PL2]
1622
```



```
R = Flatten[Table[Table[av[[t]], {12}], {t, 1, Length[av]}]];
R = R[[Length[R] - Length[xx] + 1 ;; Length[R]]];
ListPlot[R, Frame → True, FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "global temp. (K)"}]
```



Modeling

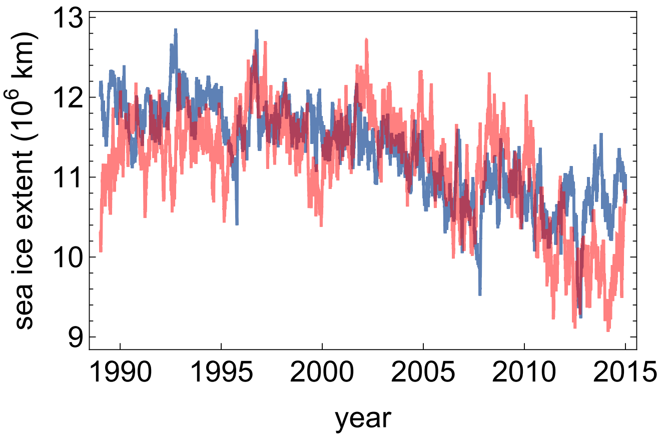
```
OU = xx[[3000 ;; 6000]];
ΔOU = Drop[OU, 1] - Drop[OU, -1];
ΔOU = ΔOU - Mean[ΔOU];
OU = FoldList[Plus, 0, ΔOU];
OU = OU - Mean[OU];
OU = Thread[{#1, #2} &[Range[Length[OU]], OU]];
est = EstimatedProcess[OU, OrnsteinUhlenbeckProcess[μ, Σ, θ]]
```

```
OrnsteinUhlenbeckProcess[0.0000320507, 0.26038, 0.0182579]
```

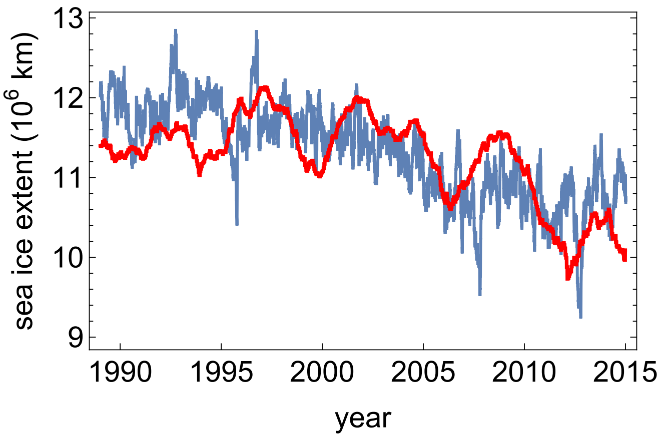
```
(*0.14 is the mean r-value*)
Clear[a]
sol = Solve[a Sqrt[1 - 4 * 0.14] == est[[3]], a]
```

```
{{a → 0.0275248}}
```

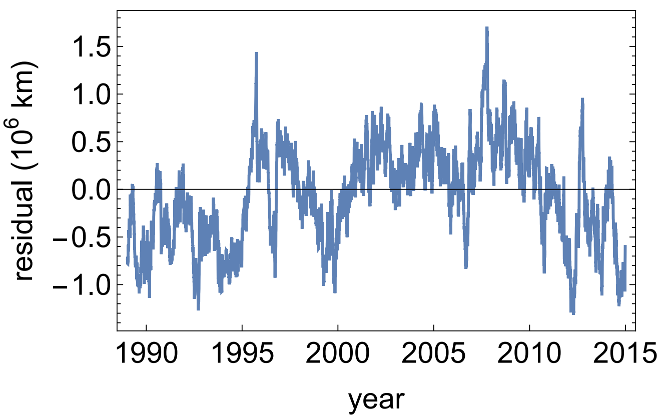
```
a = 0.02752;
Clear[t];
σ = 0.02;
r0 = 0.1;
v = 0.1;
A = 12.3;
B = 8;
r = r0;
tab = {0.25};
rliste = {r};
pliste = {};
driver = (R - First[R]) / (Last[R] - First[R]);
Monitor[
 Do[
   rand = RandomReal[NormalDistribution[0, σ], 5 * 31];
   rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
   ifun = Interpolation[rand];
   (* xxxxxxxx *)
   r = r0 + v * driver[[30 * t]];
   rliste = Append[rliste, r];
   Q0 = Last[tab];
   s = NDSolve[{Q'[tt] == a * (r - Sqrt[(1 - Q[tt]) ^2] * Q[tt]) + ifun[tt + 1],
      Q[0] == Q0}, Q, {tt, 0, 30}];
   mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
   tab = Join[tab, mid];
   , {t, 1, 316}];
 , t]
icemodel = -tab;

Q2 = ListPlot[A + B * (icemodel), Joined → True,
   PlotRange → All, PlotStyle → {Red, Opacity[0.5]}];
modelexample = A + B * (icemodel);
Show[{Q1, Q2}, PlotRange → All]
```



```
Clear[r]
r0 = 0.1;
v = 0.1;
A = 12.3;
B = 8;
fix[r_] := 1/2 (1 - Sqrt[1 - 4 r]);
Q3 = ListPlot[A - B * Map[fix[#] &, r0 + v * driver[[Range[30 * 316]]]],
    PlotStyle → {Red, Thick}, Joined → True];
Show[{Q1, Q3}, PlotRange -> All]
```



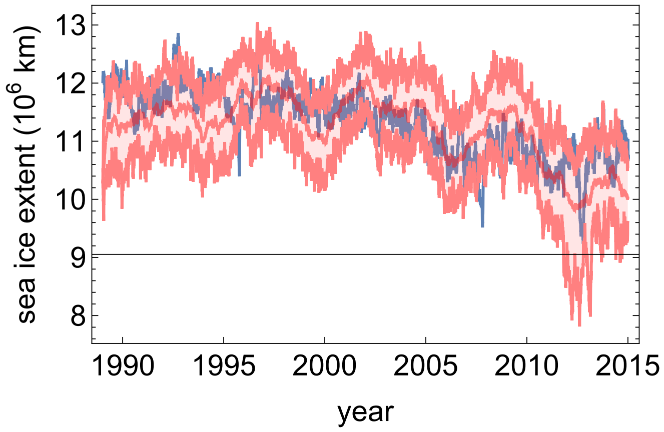

```
residual = (A - B * Map[fix[#] &, r0 + v * driver[[Range[30 * 316]]]]) - Drop[xx, 10];
ListPlot[residual, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "residual (10^6 km)"}]
```



```
mlist = {};
Monitor[
  Do[
    Clear[t];
    r = r0;
    tab = {0.25};
    rliste = {r};
    pliste = {};
    driver = (R - First[R]) / (Last[R] - First[R]);
    Monitor[
     Do[
       rand = RandomReal[NormalDistribution[0, σ], 5 * 31];
       rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
       ifun = Interpolation[rand];
       (* xxxxxxxx *)
       r = r0 + v * driver[[30 * t]];
       rliste = Append[rliste, r];
       Q0 = Last[tab];
       s = NDSolve[{Q'[tt] == a * (r - Sqrt[(1 - Q[tt])^2] * Q[tt]) + ifun[tt + 1],
           Q[0] == Q0}, Q, {tt, 0, 30}];
       mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
       tab = Join[tab, mid];
       , {t, 1, 316}];
     , t];
    icemodel = -tab;
    model = A + B * (icemodel);
    mlist = Append[mlist, model];
    , {j, 1, 20}];
  , j];
```

```
mean = Map[Mean[#] &, Transpose[mlist]];
low = Map[Quantile[#, 0.025] &, Transpose[mlist]];
high = Map[Quantile[#, 1 - 0.025] &, Transpose[mlist]];
```

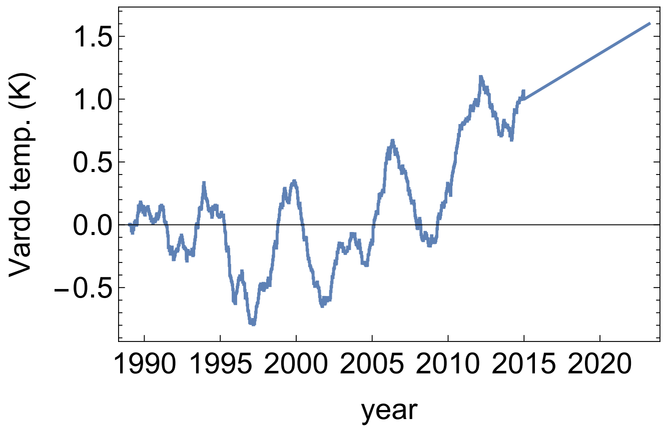
```
Qmean = ListPlot[mean, Joined → True,
   PlotRange → All, PlotStyle → {Red, Opacity[0.5]}];
Qampl = ListPlot[{low, high}, Joined → True, PlotRange → All,
   PlotStyle → {Pink}, Filling → {1 → {2}}];
Q11 = ListPlot[xx, Frame → True, FrameTicks →
    {{Automatic, Automatic}, {ticks, None}}, FrameStyle → Directive[16],
   Joined → True, FrameLabel → {"year", "sea ice extent (10^6 km)"}];
Show[{Q11, Qmean, Qampl}, PlotRange → All]
```



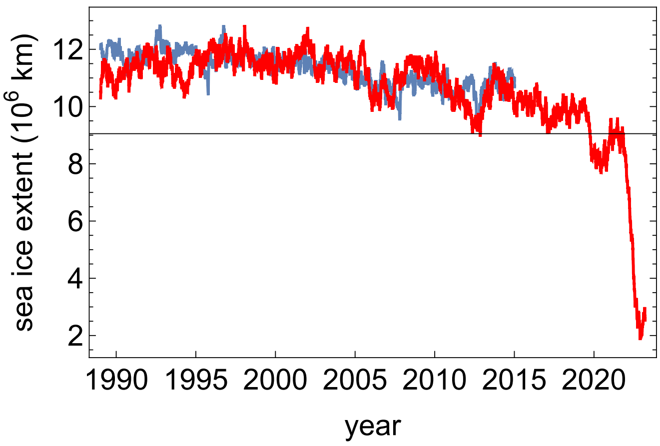
Future prediction

Scenario for temperature:

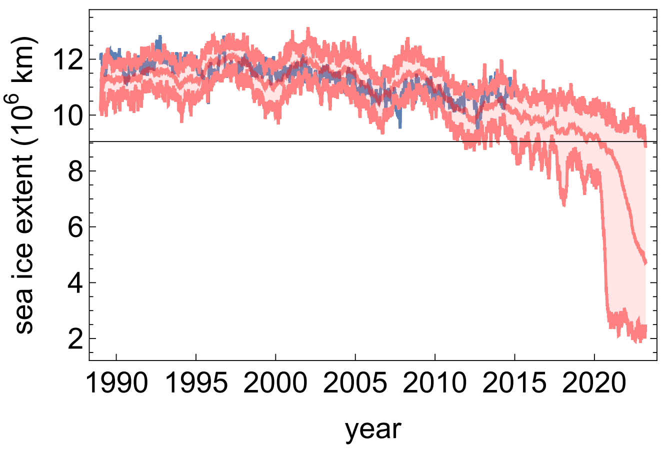
```
ticks = {{359, "1990"}, {359 + 5 * 365, "1995"},
   {359 + 10 * 365, "2000"}, {359 + 15 * 365, "2005"}, {359 + 20 * 365, "2010"},
   {359 + 25 * 365, "2015"}, {359 + 30 * 365, "2020"}};
newdriver = Join[driver, Last[driver] + 0.0002 * Range[3000]];
ListPlot[newdriver, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "Vardo temp. (K)"}]
```



```
Clear[t];
r = r0;
tab = {0.25};
rliste = {r};
pliste = {};
driver = (R - First[R]) / (Last[R] - First[R]);
Monitor[
 Do[
   rand = RandomReal[NormalDistribution[0, σ], 5 * 31];
   rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
   ifun = Interpolation[rand];
   (* xxxxxxxx *)
   r = r0 + v * newdriver[[30 * t]];
   rliste = Append[rliste, r];
   Q0 = Last[tab];
   s = NDSolve[{Q'[tt] == a * (r - Sqrt[(1 - Q[tt]) ^2] * Q[tt]) + ifun[tt + 1],
      Q[0] == Q0}, Q, {tt, 0, 30}];
   mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
   tab = Join[tab, mid];
   , {t, 1, 316 + 100}];
 , t]
icemodel = -tab;
Q3 =
  ListPlot[A + B * (icemodel), Joined → True, PlotRange → All, PlotStyle → Red];

Show[{Q1, Q3}, PlotRange → All,
 FrameTicks → {{Automatic, Automatic}, {ticks, None}}]
```



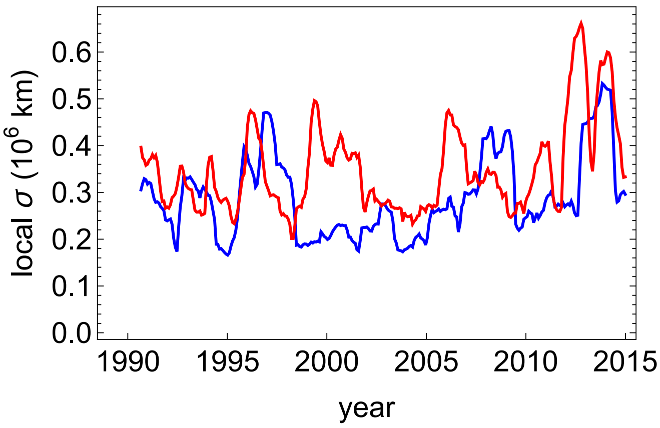
```
mlist = {};
Monitor[
  Do[
    Clear[t];
    r = r0;
    tab = {0.25};
    rliste = {r};
    pliste = {};
    driver = (R - First[R]) / (Last[R] - First[R]);
    Monitor[
     Do[
       rand = RandomReal[NormalDistribution[0, σ], 5 * 31];
       rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
       ifun = Interpolation[rand];
       (* xxxxxxxx *)
       r = r0 + v * newdriver[[30 * t]];
       rliste = Append[rliste, r];
       Q0 = Last[tab];
       s = NDSolve[{Q'[tt] == a * (r - Sqrt[(1 - Q[tt])^2] * Q[tt]) + ifun[tt + 1],
           Q[0] == Q0}, Q, {tt, 0, 30}];
       mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
       tab = Join[tab, mid];
       , {t, 1, 316 + 100}];
     , t];
    icemodel = -tab;
    model = A + B * (icemodel);
    mlist = Append[mlist, model];
    , {j, 1, 20}];
  , j];

mean = Map[Mean[#] &, Transpose[mlist]];
low = Map[Quantile[#, 0.025] &, Transpose[mlist]];
high = Map[Quantile[#, 1 - 0.025] &, Transpose[mlist]];


Qmean = ListPlot[mean, Joined → True,
    PlotRange → All, PlotStyle → {Red, Opacity[0.5]}];
Qampl = ListPlot[{low, high}, Joined → True, PlotRange → All,
    PlotStyle → {Pink}, Filling → {1 → {2}}];
Q11 = ListPlot[xx, Frame → True, FrameTicks →
     {{Automatic, Automatic}, {ticks, None}}, FrameStyle → Directive[16],
    Joined → True, FrameLabel → {"year", "sea ice extent (10^6 km)"}];
Show[{Q11, Qmean, Qampl}, PlotRange → All]
```



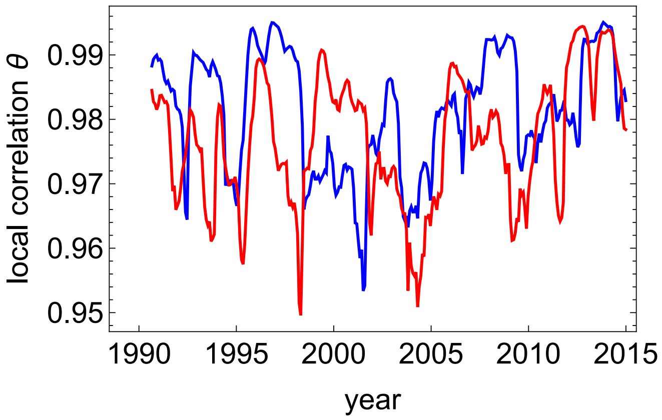
Early warning indications

Estimate variance and correlation. Red for model. Blue for real data.

```
win = 600;
σlist = {};
τlist = {};
Monitor[
  Do[
    qqq = modelexample[[t - win ;; t]];
    σloc = StandardDeviation[qqq]; (*local standard deviation*)
    τloc = Correlation[Drop[qqq, 1], Drop[qqq, -1]];
    (*local correlation scale*)
    σlist = Append[σlist, {t, σloc}];
    τlist = Append[τlist, {t, τloc}];
    , {t, win + 1, Length[modelexample], 30}];
  , t];
QQ1 = ListPlot[σlist, Frame → True,
   FrameTicks → {{Automatic, Automatic}, {ticks, None}},
   FrameStyle → Directive[16], Joined → True,
   FrameLabel → {"year", "local σ (10^6 km)"}, PlotStyle → Red];
QQ2 = ListPlot[τlist, Frame → True, FrameTicks →
    {{Automatic, Automatic}, {ticks, None}}, FrameStyle → Directive[16],
   Joined → True, FrameLabel → {"year", "local correlation θ"}, PlotStyle → Red];
win = 600;
σlist = {};
τlist = {};
Monitor[
  Do[
    qqq = xx[[t - win ;; t]];
    σloc = StandardDeviation[qqq]; (*local standard deviation*)
    τloc = Correlation[Drop[qqq, 1], Drop[qqq, -1]];
    (*local correlation scale*)
    σlist = Append[σlist, {t, σloc}];
    τlist = Append[τlist, {t, τloc}];
    , {t, win + 1, Length[modelexample], 30}];
  , t];
QQ3 = ListPlot[σlist, Frame → True,
   FrameTicks → {{Automatic, Automatic}, {ticks, None}},
   FrameStyle → Directive[16], Joined → True,
   FrameLabel → {"year", "local σ (10^6 km)"}, PlotStyle → Blue];
QQ4 = ListPlot[τlist, Frame → True,
   FrameTicks → {{Automatic, Automatic}, {ticks, None}},
   FrameStyle → Directive[16], Joined → True,
   FrameLabel → {"year", "local correlation θ"}, PlotStyle → Blue];
Show[{QQ3, QQ1}, PlotRange → All]
```



```
Show[{QQ4, QQ2}, PlotRange → All]
```

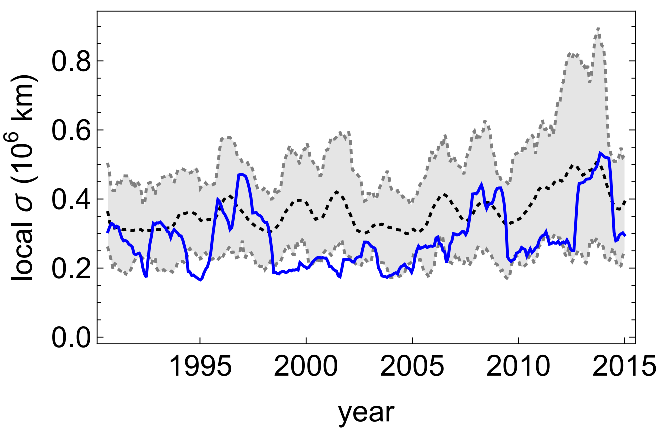


Monte Carlo

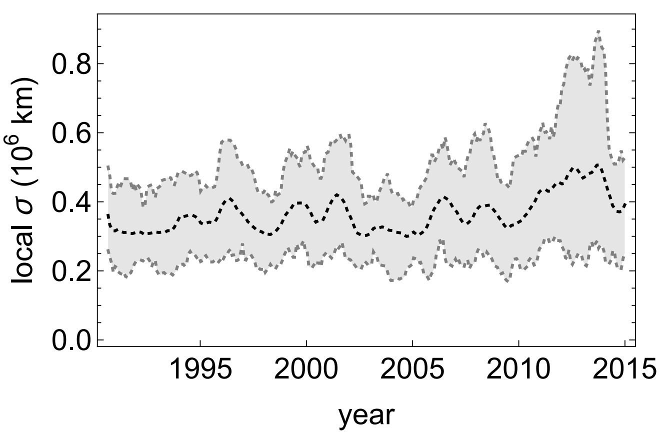
```
σlistlist = {};
τlistlist = {};
```

```
Monitor[
 Do[
   Clear[t];
   r = r0;
   tab = {0.25};
   rliste = {r};
   pliste = {};
   driver = (R - First[R]) / (Last[R] - First[R]);
   Monitor[
     Do[
       rand = RandomReal[NormalDistribution[0, σ], 5 * 31];
       rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
       ifun = Interpolation[rand];
       (* xxxxxxxx *)
       r = r0 + v * driver[[30 * t]];
       rliste = Append[rliste, r];
       Q0 = Last[tab];
       s = NDSolve[{Q'[tt] == a * (r - Sqrt[(1 - Q[tt])^2] * Q[tt]) + ifun[tt + 1],
           Q[0] == Q0}, Q, {tt, 0, 30}];
       mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
       tab = Join[tab, mid];
       , {t, 1, 316}];
    , t]
    icemodel = -tab;
   modelexample = A + B * (icemodel);
   win = 600;
   σlist = {};
   τlist = {};
   Do[
    qqq = modelexample[[t - win ;; t]];
    σloc = StandardDeviation[qqq]; (*local standard deviation*)
    τloc = Correlation[Drop[qqq, 1], Drop[qqq, -1]];
    (*local correlation scale*)
    σlist = Append[σlist, {t, σloc}];
    τlist = Append[τlist, {t, τloc}];
    , {t, win + 1, Length[modelexample], 30}];
   σlistlist = Append[σlistlist, σlist];
   τlistlist = Append[τlistlist, τlist];
   , {run, 1, 20}];
  , {run, t}]
```

```
times = σlistlist[[1]][[All, 1]];
ticks = {{359, "1990"}, {359 + 5 * 365, "1995"},
   {359 + 10 * 365, "2000"}, {359 + 15 * 365, "2005"}, {359 + 20 * 365, "2010"},
   {359 + 25 * 365, "2015"}, {359 + 30 * 365, "2020"}};
σmean = Map[Mean[#] &, Transpose[Map[#[[All, 2]] &, σlistlist]]];
σlow = Map[Quantile[#, 0.025] &, Transpose[Map[#[[All, 2]] &, σlistlist]]];
σhigh = Map[Quantile[#, 1 - 0.025] &, Transpose[Map[#[[All, 2]] &, σlistlist]]];

σmean = Thread[{#1, #2} &[times, σmean]];
σlow = Thread[{#1, #2} &[times, σlow]];
σhigh = Thread[{#1, #2} &[times, σhigh]];

PL1 = ListPlot[σmean, Frame → True,
   FrameTicks → {{Automatic, Automatic}, {ticks, None}},
   FrameStyle → Directive[16], Joined → True,
   FrameLabel → {"year", "local correlation θ"}, PlotStyle → {Black, Dotted}];
PL2 = ListPlot[{σlow, σhigh}, Frame → True, FrameTicks →
    {{Automatic, Automatic}, {ticks, None}}, FrameStyle → Directive[16],
   Joined → True, FrameLabel → {"year", "local σ (10^6 km)"},
   PlotStyle → {{Dotted, Gray}, {Dotted, Gray}},
   Filling → {1 → {2}}, PlotRange → All];
PL3 = Show[PL2, PL1, QQ3]
```




```
Show[PL2, PL1]
```



Appendix I

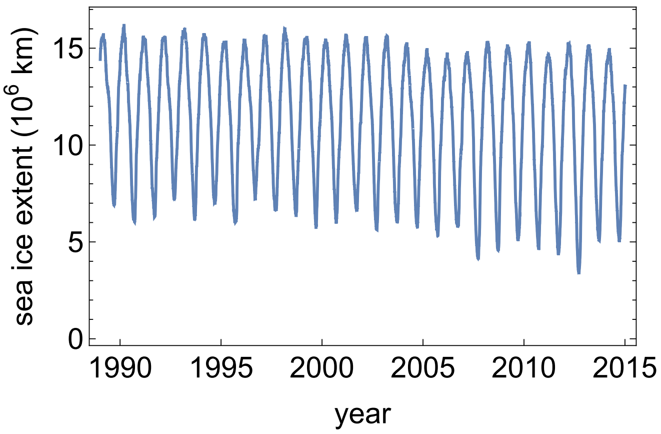
Vardø temperature and Brownian motion

The following code contains all the calculations and plots for real data and modeling of future predictions of sea ice extent in Arctic using temperature in Vardø as a driver and Brownian motion as an additive noise, discussed in section "Brownian motion" in the chapter "Arctic temperature". The code was written in the program "Mathematica".

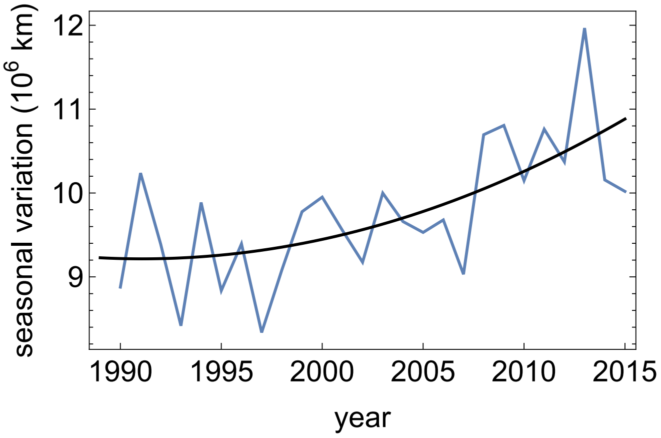
Reading and preparing data:

Read sea ice data:

```
x = ReadList[
    "Dropbox/Master Theses/sea ice/NH_seaice_extent_final.csv", String];
x = Table[ToExpression[StringSplit[x[[t]], ","][[4]]], {t, 3, Length[x]}];
ticks = {{359, "1990"}, {359 + 5 * 365, "1995"}, {359 + 10 * 365, "2000"},
    {359 + 15 * 365, "2005"}, {359 + 20 * 365, "2010"}, {359 + 25 * 365, "2015"}};
y = Drop[x, 2096];
ListPlot[y, Frame → True, FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "sea ice extent (10^6 km)"}]
```

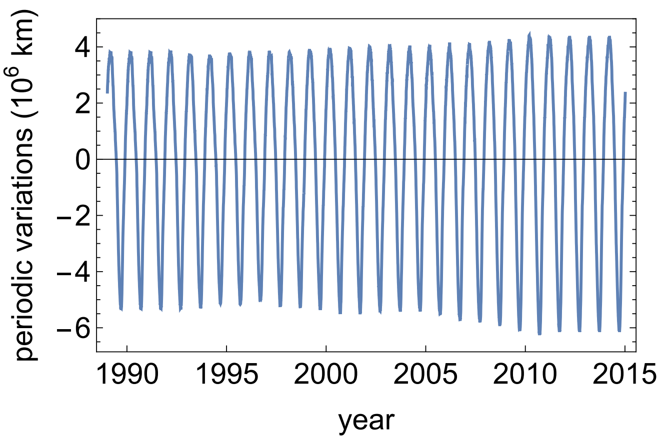



```
amplitudes = Map[Max[#] - Min[#] &, Partition[y, 365]];
fit = Fit[amplitudes, {zz^2, zz, 1}, zz];
PL1 = ListPlot[amplitudes, Joined → True];
PL2 = Plot[fit, {zz, 0, 26}, PlotStyle → Black];
newticks = {{1 + 0, "1990"}, {1 + 5, "1995"},
    {1 + 10, "2000"}, {1 + 15, "2005"}, {1 + 20, "2010"}, {1 + 25, "2015"}};
Show[{PL1, PL2}, Frame → True, FrameTicks →
   {{Automatic, Automatic}, {newticks, None}}, FrameStyle → Directive[16],
 FrameLabel → {"year", "seasonal variation (10^6 km)"}]
```



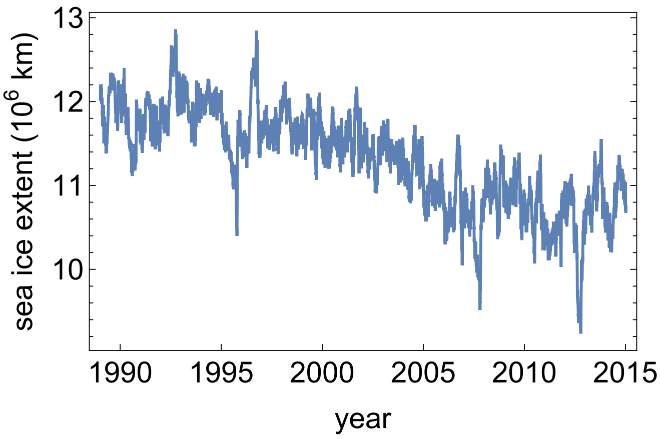
```
clim1 = Map[Mean[#] &, Transpose[Partition[y, 365][[1 ;; 5]]]];
clim2 = Map[Mean[#] &, Transpose[Partition[y, 365][[1 ;; 5]]]];
clim3 = Map[Mean[#] &, Transpose[Partition[y, 365][[1 ;; 5]]]];
clim4 = Map[Mean[#] &, Transpose[Partition[y, 365][[2 ;; 6]]]];
clim5 = Map[Mean[#] &, Transpose[Partition[y, 365][[3 ;; 7]]]];
clim6 = Map[Mean[#] &, Transpose[Partition[y, 365][[4 ;; 8]]]];
clim7 = Map[Mean[#] &, Transpose[Partition[y, 365][[5 ;; 9]]]];
clim8 = Map[Mean[#] &, Transpose[Partition[y, 365][[6 ;; 10]]]];
clim9 = Map[Mean[#] &, Transpose[Partition[y, 365][[7 ;; 11]]]];
clim10 = Map[Mean[#] &, Transpose[Partition[y, 365][[8 ;; 12]]]];
clim11 = Map[Mean[#] &, Transpose[Partition[y, 365][[9 ;; 13]]]];
clim12 = Map[Mean[#] &, Transpose[Partition[y, 365][[10 ;; 14]]]];
clim13 = Map[Mean[#] &, Transpose[Partition[y, 365][[11 ;; 15]]]];
clim14 = Map[Mean[#] &, Transpose[Partition[y, 365][[12 ;; 16]]]];
clim15 = Map[Mean[#] &, Transpose[Partition[y, 365][[13 ;; 17]]]];
clim16 = Map[Mean[#] &, Transpose[Partition[y, 365][[14 ;; 18]]]];
clim17 = Map[Mean[#] &, Transpose[Partition[y, 365][[15 ;; 19]]]];
clim18 = Map[Mean[#] &, Transpose[Partition[y, 365][[16 ;; 20]]]];
clim19 = Map[Mean[#] &, Transpose[Partition[y, 365][[17 ;; 21]]]];
clim20 = Map[Mean[#] &, Transpose[Partition[y, 365][[18 ;; 22]]]];
clim21 = Map[Mean[#] &, Transpose[Partition[y, 365][[19 ;; 23]]]];
clim22 = Map[Mean[#] &, Transpose[Partition[y, 365][[20 ;; 24]]]];
clim23 = Map[Mean[#] &, Transpose[Partition[y, 365][[21 ;; 25]]]];
clim24 = Map[Mean[#] &, Transpose[Partition[y, 365][[22 ;; 26]]]];
clim25 = Map[Mean[#] &, Transpose[Partition[y, 365][[22 ;; 26]]]];
clim26 = Map[Mean[#] &, Transpose[Partition[y, 365][[22 ;; 26]]]];
climtab = {clim1, clim2, clim3, clim4, clim5, clim6, clim7, clim8, clim9,
   clim10, clim11, clim12, clim13, clim14, clim15, clim16, clim17, clim18,
   clim19, clim20, clim21, clim22, clim23, clim24, clim25, clim26};
climtab = Map[# - Mean[#] &, climtab];
factors = Map[Max[#] - Min[#] &, climtab];
climseries = Flatten[climtab];
```

```
ListPlot[climseries, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "periodic variations (10^6 km)"}]
```



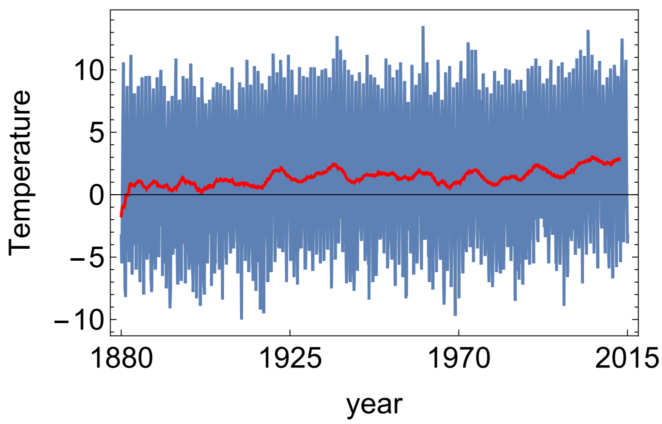
Remove climatology:

```
xx = y - climseries;
years = Partition[xx, 365];
yearmeans = Map[Mean[#] &, years];
years = Table[(years[[i]] - yearmeans[[i]]) * (factors[[i]] / Mean[factors]) +
     yearmeans[[i]], {i, 1, 26}];
xx = Flatten[years];
Q1 = ListPlot[xx, Frame → True,
   FrameTicks → {{Automatic, Automatic}, {ticks, None}},
   FrameStyle → Directive[16], Joined → True,
   FrameLabel → {"year", "sea ice extent (10^6 km)"}]
```

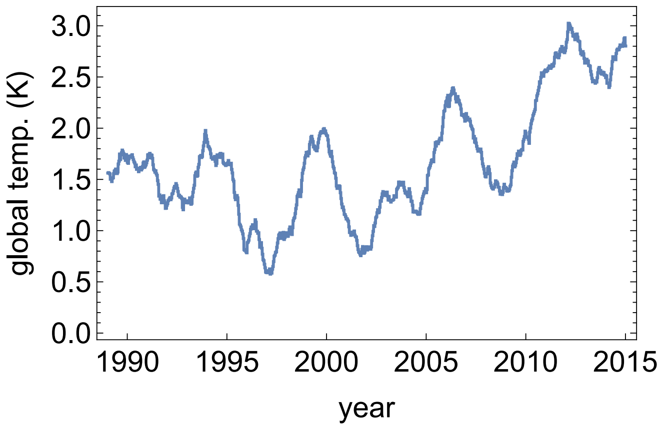


Vardo temperature anomaly:

```
T = ReadList["Dropbox/Master Theses/Vardo_temperature/Vardotemp.txt"];
ticks2 =
  {{1, "1880"}, {1 + 12 * 45, "1925"}, {1 + 12 * 90, "1970"}, {1 + 12 * 135, "2015"}};
PL1 = ListPlot[T, Frame → True, FrameTicks →
     {{Automatic, Automatic}, {ticks2, None}}, FrameStyle → Directive[16],
    Joined → True, FrameLabel → {"year", "Temperature"}];
av = Drop[MovingAverage[ArrayPad[T, 5 * 12 / 2, "Fixed"], 5 * 12], -28];
PL2 = ListPlot[av, PlotStyle → Red, Joined → True];
Length[T]
Show[PL1, PL2]
1622
```



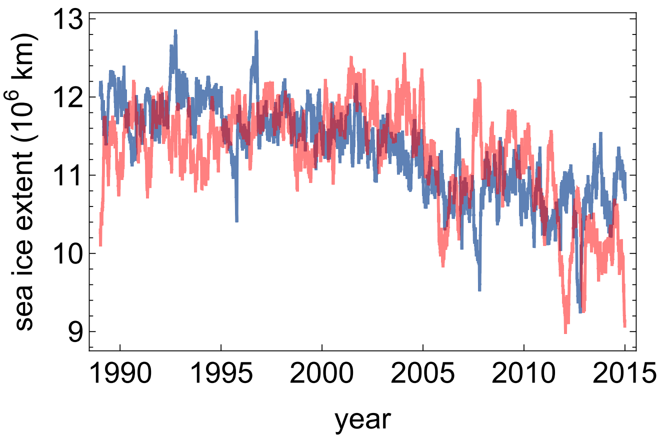
```
R = Flatten[Table[Table[av[[t]], {12}], {t, 1, Length[av]}]];
R = R[[Length[R] - Length[xx] + 1 ;; Length[R]]];
ListPlot[R, Frame → True, FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "global temp. (K)"}]
```



Modeling

```
OU = xx[[3000 ;; 6000]];
ΔOU = Drop[OU, 1] - Drop[OU, -1];
ΔOU = ΔOU - Mean[ΔOU];
OU = FoldList[Plus, 0, ΔOU];
OU = OU - Mean[OU];
OU = Thread[{#1, #2} &[Range[Length[OU]], OU]];
est = EstimatedProcess[OU, OrnsteinUhlenbeckProcess[μ, Σ, θ]]

OrnsteinUhlenbeckProcess[0.0000320507, 0.26038, 0.0182579]

(*0.14 is the mean r-value*)
Clear[a]
sol = Solve[a Sqrt[1 - 4 * 0.14] == est[[3]], a]
{{a → 0.0275248}}
```

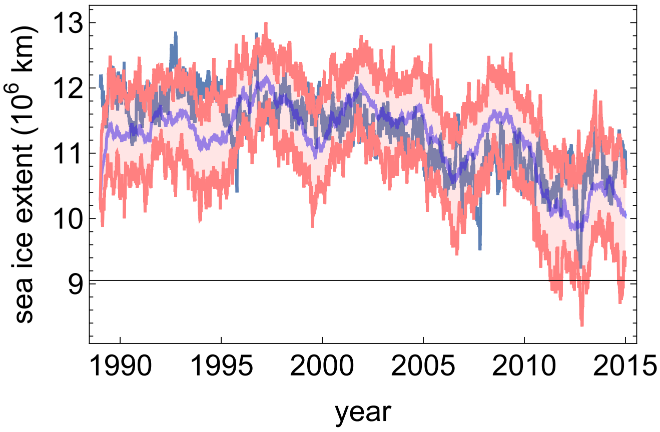
```
a = 0.02752;
Clear[t];
σ = 0.01;
r0 = 0.1;
v = 0.1;
A = 12.3;
B = 8;
r = r0;
tab = {0.25};
rliste = {r};
pliste = {};
driver = (R - First[R]) / (Last[R] - First[R]);
Monitor[
 Do[
   rand =
    RandomFunction[FractionalBrownianMotionProcess[.64], {0, 5 * 31 + 1, 1}];
   rand = Drop[rand["Path"][[All, 2]], 1];
   rand = Drop[rand, 1] - Drop[rand, -1];
   rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
   ifun = Interpolation[rand];
   (* xxxxxxxx *)
   r = r0 + v * driver[[30 * t]];
   rliste = Append[rliste, r];
   Q0 = Last[tab];
   s = NDSolve[{Q'[tt] == a * (r - Sqrt[(1 - Q[tt])^2] * Q[tt]) + σ * ifun[tt + 1],
      Q[0] == Q0}, Q, {tt, 0, 30}];
   mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
   tab = Join[tab, mid];
   , {t, 1, 316}];
 , t]
icemodel = -tab;

Q2 = ListPlot[A + B * (icemodel), Joined → True,
   PlotRange → All, PlotStyle → {Red, Opacity[0.5]}];
modelexample = A + B * (icemodel);
Show[{Q1, Q2}, PlotRange → All]
```



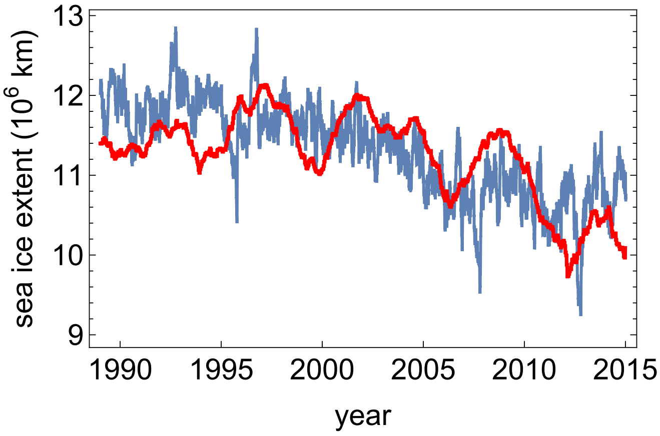
```
mlist = {};
Monitor[
  Do[
    Clear[t];
    r = r0;
    tab = {0.25};
    rliste = {r};
    pliste = {};
    driver = (R - First[R]) / (Last[R] - First[R]);
    Monitor[
     Do[
       rand =
        RandomFunction[FractionalBrownianMotionProcess[.64], {0, 5 * 31 + 1, 1}];
       rand = Drop[rand["Path"][[All, 2]], 1];
       rand = Drop[rand, 1] - Drop[rand, -1];
       rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
       ifun = Interpolation[rand];
       (* xxxxxxxx *)
       r = r0 + v * driver[[30 * t]];
       rliste = Append[rliste, r];
       Q0 = Last[tab];
       s = NDSolve[{Q'[tt] == a * (r - Sqrt[(1 - Q[tt])^2] * Q[tt]) + σ * ifun[tt + 1],
          Q[0] == Q0}, Q, {tt, 0, 30}];
       mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
       tab = Join[tab, mid];
       , {t, 1, 316}];
     , t];
    icemodel = -tab;
    model = A + B * (icemodel);
    mlist = Append[mlist, model];
    , {j, 1, 20}];
  , j];
```

```
mean = Map[Mean[#] &, Transpose[mlist]];
low = Map[Quantile[#, 0.025] &, Transpose[mlist]];
high = Map[Quantile[#, 1 - 0.025] &, Transpose[mlist]];
```

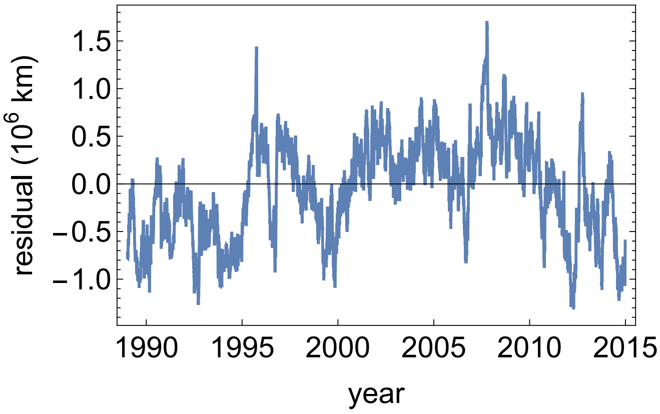
```
Qmean = ListPlot[mean, Joined → True,
   PlotRange → All, PlotStyle → {Blue, Opacity[0.5]}];
Qampl = ListPlot[{low, high}, Joined → True, PlotRange → All,
   PlotStyle → {Pink}, Filling → {1 → {2}}];
Q11 = ListPlot[xx, Frame → True, FrameTicks →
    {{Automatic, Automatic}, {ticks, None}}, FrameStyle → Directive[16],
   Joined → True, FrameLabel → {"year", "sea ice extent (10^6 km)"}];
Show[{Q11, Qmean, Qampl}, PlotRange → All]
```




```
Clear[r]
r0 = 0.1;
v = 0.1;
A = 12.3;
B = 8;
fix[r_] := 1/2 (1 - √(1 - 4 r));
Q3 = ListPlot[A - B * Map[fix[#] &, r0 + v * driver[[Range[30 * 316]]]],
   PlotStyle → {Red, Thick}, Joined → True];
Show[{Q1, Q3}, PlotRange -> All]
```



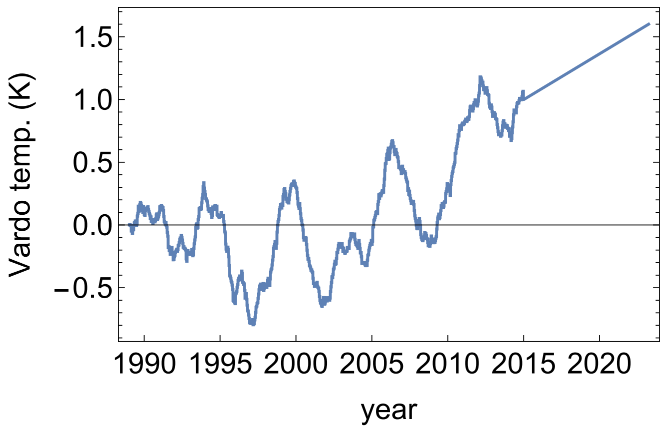
```
residual = (A - B * Map[fix[#] &, r0 + v * driver[[Range[30 * 316]]]]) - Drop[xx, 10];
ListPlot[residual, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "residual  (10^6 km)"}]
```



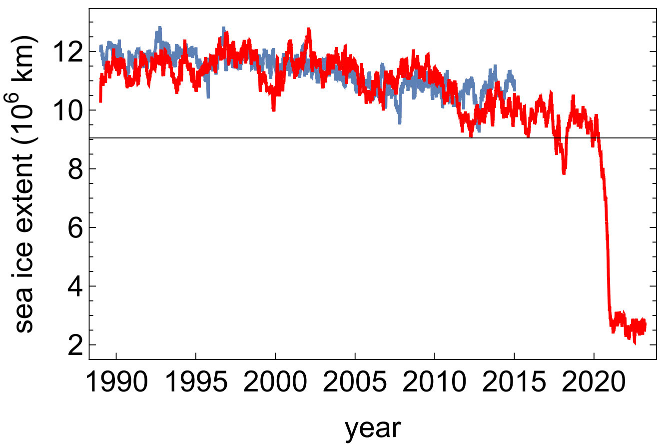
Future prediction

Scenario for temperature:

```
ticks = {{359, "1990"}, {359 + 5 * 365, "1995"},
   {359 + 10 * 365, "2000"}, {359 + 15 * 365, "2005"}, {359 + 20 * 365, "2010"},
   {359 + 25 * 365, "2015"}, {359 + 30 * 365, "2020"}};
newdriver = Join[driver, Last[driver] + 0.0002 * Range[3000]];
ListPlot[newdriver, Frame → True,
 FrameTicks → {{Automatic, Automatic}, {ticks, None}},
 FrameStyle → Directive[16], Joined → True,
 FrameLabel → {"year", "Vardo temp.  (K)"}]
```

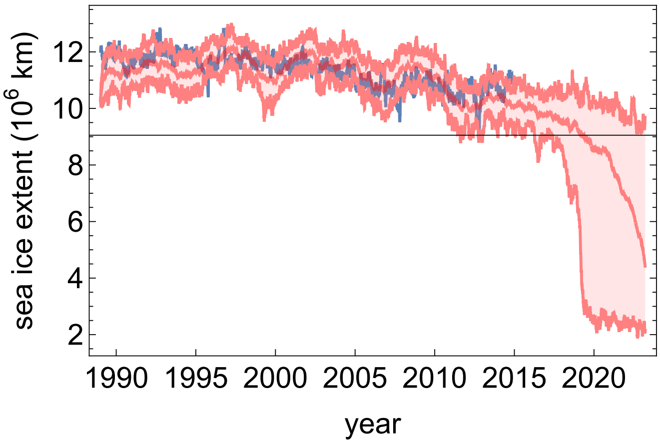


```
Clear[t];
r = r0;
tab = {0.25};
rliste = {r};
pliste = {};
driver = (R - First[R]) / (Last[R] - First[R]);
Monitor[
 Do[
   rand =
    RandomFunction[FractionalBrownianMotionProcess[.64], {0, 5 * 31 + 1, 1}];
   rand = Drop[rand["Path"][[All, 2]], 1];
   rand = Drop[rand, 1] - Drop[rand, -1];
   rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
   ifun = Interpolation[rand];
   (* xxxxxxxx *)
   r = r0 + v * newdriver[[30 * t]];
   rliste = Append[rliste, r];
   Q0 = Last[tab];
   s = NDSolve[{Q'[tt] == a * (r - Sqrt[(1 - Q[tt])^2] * Q[tt]) + σ * ifun[tt + 1],
      Q[0] == Q0}, Q, {tt, 0, 30}];
   mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
   tab = Join[tab, mid];
   , {t, 1, 316 + 100}];
 , t]
icemodel = -tab;
Q3 =
  ListPlot[A + B * (icemodel), Joined → True, PlotRange → All, PlotStyle → Red];

Show[{Q1, Q3}, PlotRange → All,
 FrameTicks → {{Automatic, Automatic}, {ticks, None}}]
```



```
mlist = {};
Monitor[
  Do[
    Clear[t];
    r = r0;
    tab = {0.25};
    rliste = {r};
    pliste = {};
    driver = (R - First[R]) / (Last[R] - First[R]);
    Monitor[
     Do[
       rand =
        RandomFunction[FractionalBrownianMotionProcess[.64], {0, 5 * 31 + 1, 1}];
       rand = Drop[rand["Path"][[All, 2]], 1];
       rand = Drop[rand, 1] - Drop[rand, -1];
       rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
       ifun = Interpolation[rand];
       (* xxxxxxxx *)
       r = r0 + v * newdriver[[30 * t]];
       rliste = Append[rliste, r];
       Q0 = Last[tab];
       s = NDSolve[{Q'[tt] == a * (r - Sqrt[(1 - Q[tt])^2] * Q[tt]) + σ * ifun[tt + 1],
          Q[0] == Q0}, Q, {tt, 0, 30}];
       mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
       tab = Join[tab, mid];
       , {t, 1, 316 + 100}];
     , t];
    icemodel = -tab;
    model = A + B * (icemodel);
    mlist = Append[mlist, model];
    , {j, 1, 20}];
  , j];
```

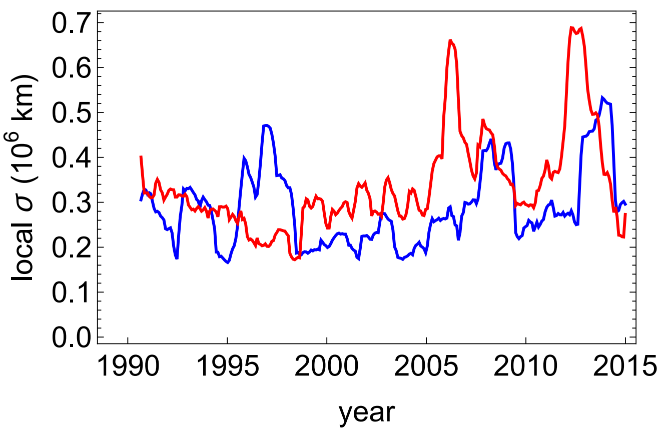
```
mean = Map[Mean[#] &, Transpose[mlist]];
low = Map[Quantile[#, 0.025] &, Transpose[mlist]];
high = Map[Quantile[#, 1 - 0.025] &, Transpose[mlist]];


Qmean = ListPlot[mean, Joined → True,
   PlotRange → All, PlotStyle → {Red, Opacity[0.5]}];
Qampl = ListPlot[{low, high}, Joined → True, PlotRange → All,
   PlotStyle → {Pink}, Filling → {1 → {2}}];
Q11 = ListPlot[xx, Frame → True, FrameTicks →
    {{Automatic, Automatic}, {ticks, None}}, FrameStyle → Directive[16],
   Joined → True, FrameLabel → {"year", "sea ice extent (10^6 km)"}];
Show[{Q11, Qmean, Qampl}, PlotRange → All]
```



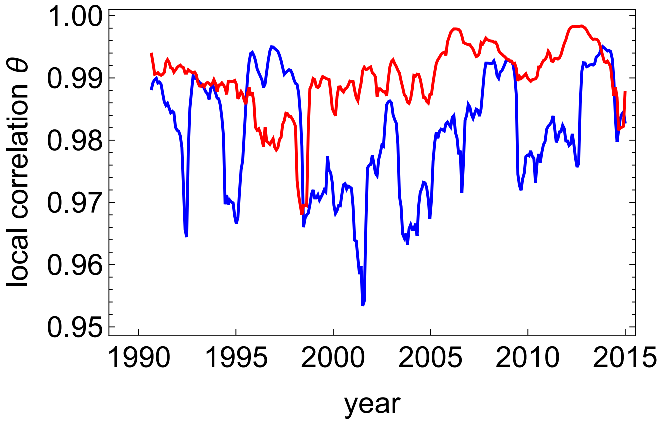
Early warning indications

Estimate variance and correlation. Red for model. Blue for real data.

```
win = 600;
σlist = {};
τlist = {};
Monitor[
  Do[
    qqq = modelexample[[t - win ;; t]];
    σloc = StandardDeviation[qqq]; (*local standard deviation*)
    τloc = Correlation[Drop[qqq, 1], Drop[qqq, -1]];
    (*local correlation scale*)
    σlist = Append[σlist, {t, σloc}];
    τlist = Append[τlist, {t, τloc}];
    , {t, win + 1, Length[modelexample], 30}];
  , t];
QQ1 = ListPlot[σlist, Frame → True,
   FrameTicks → {{Automatic, Automatic}, {ticks, None}},
   FrameStyle → Directive[16], Joined → True,
   FrameLabel → {"year", "local σ (10^6 km)"}, PlotStyle → Red];
QQ2 = ListPlot[τlist, Frame → True, FrameTicks →
    {{Automatic, Automatic}, {ticks, None}}, FrameStyle → Directive[16],
   Joined → True, FrameLabel → {"year", "local correlation θ"}, PlotStyle → Red];
win = 600;
σlist = {};
τlist = {};
Monitor[
  Do[
    qqq = xx[[t - win ;; t]];
    σloc = StandardDeviation[qqq]; (*local standard deviation*)
    τloc = Correlation[Drop[qqq, 1], Drop[qqq, -1]];
    (*local correlation scale*)
    σlist = Append[σlist, {t, σloc}];
    τlist = Append[τlist, {t, τloc}];
    , {t, win + 1, Length[modelexample], 30}];
  , t];
QQ3 = ListPlot[σlist, Frame → True,
   FrameTicks → {{Automatic, Automatic}, {ticks, None}},
   FrameStyle → Directive[16], Joined → True,
   FrameLabel → {"year", "local σ (10^6 km)"}, PlotStyle → Blue];
QQ4 = ListPlot[τlist, Frame → True,
   FrameTicks → {{Automatic, Automatic}, {ticks, None}},
   FrameStyle → Directive[16], Joined → True,
   FrameLabel → {"year", "local correlation θ"}, PlotStyle → Blue];
Show[{QQ3, QQ1}, PlotRange → All]
```



```
Show[{QQ4, QQ2}, PlotRange → All]
```

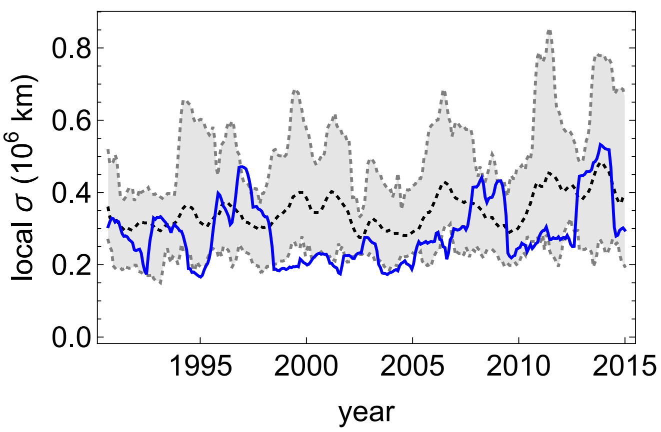


Monte Carlo

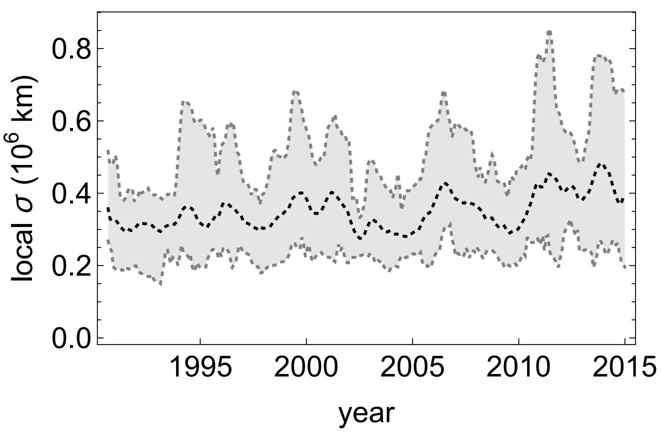
```
σlistlist = {};
τlistlist = {};
```

```
Monitor[
 Do[
   Clear[t];
   r = r0;
   tab = {0.25};
   rliste = {r};
   pliste = {};
   driver = (R - First[R]) / (Last[R] - First[R]);
   Monitor[
     Do[
       rand =
        RandomFunction[FractionalBrownianMotionProcess[.64], {0, 5 * 31 + 1, 1}];
       rand = Drop[rand["Path"][[All, 2]], 1];
       rand = Drop[rand, 1] - Drop[rand, -1];
       rand = Thread[{#1, #2} &[Range[5 * 31] / 5., rand]];
       ifun = Interpolation[rand];
       (* xxxxxxxx *)
       r = r0 + v * driver[[30 * t]];
       rliste = Append[rliste, r];
       Q0 = Last[tab];
       s = NDSolve[{Q'[tt] == a * (r - Sqrt[(1 - Q[tt])^2] * Q[tt]) + σ * ifun[tt + 1],
           Q[0] == Q0}, Q, {tt, 0, 30}];
       mid = Flatten[Evaluate[Q[tt] /. s] /. tt → Range[30]];
       tab = Join[tab, mid];
       , {t, 1, 316}];
     , t]
    icemodel = -tab;
   modelexample = A + B * (icemodel);
   win = 600;
   σlist = {};
   τlist = {};
   Do[
    qqq = modelexample[[t - win ;; t]];
    σloc = StandardDeviation[qqq]; (*local standard deviation*)
    τloc = Correlation[Drop[qqq, 1], Drop[qqq, -1]];
    (*local correlation scale*)
    σlist = Append[σlist, {t, σloc}];
    τlist = Append[τlist, {t, τloc}];
    , {t, win + 1, Length[modelexample], 30}];
   σlistlist = Append[σlistlist, σlist];
   τlistlist = Append[τlistlist, τlist];
   , {run, 1, 20}];
 , {run, t}]
```

```
times = σlistlist[[1]][[All, 1]];
ticks = {{359, "1990"}, {359 + 5 * 365, "1995"},
   {359 + 10 * 365, "2000"}, {359 + 15 * 365, "2005"}, {359 + 20 * 365, "2010"},
   {359 + 25 * 365, "2015"}, {359 + 30 * 365, "2020"}};
σmean = Map[Mean[#] &, Transpose[Map[#[[All, 2]] &, σlistlist]]];
σlow = Map[Quantile[#, 0.025] &, Transpose[Map[#[[All, 2]] &, σlistlist]]];
σhigh = Map[Quantile[#, 1 - 0.025] &, Transpose[Map[#[[All, 2]] &, σlistlist]]];

σmean = Thread[{#1, #2} &[times, σmean]];
σlow = Thread[{#1, #2} &[times, σlow]];
σhigh = Thread[{#1, #2} &[times, σhigh]];

PL1 = ListPlot[σmean, Frame → True,
   FrameTicks → {{Automatic, Automatic}, {ticks, None}},
   FrameStyle → Directive[16], Joined → True,
   FrameLabel → {"year", "local correlation θ"}, PlotStyle → {Black, Dotted}];
PL2 = ListPlot[{σlow, σhigh}, Frame → True, FrameTicks →
    {{Automatic, Automatic}, {ticks, None}}, FrameStyle → Directive[16],
   Joined → True, FrameLabel → {"year", "local σ (10^6 km)"},
   PlotStyle → {{Dotted, Gray}, {Dotted, Gray}},
   Filling → {1 → {2}}, PlotRange → All];
PL3 = Show[PL2, PL1, QQ3]
```




```
Show[PL2, PL1]
```



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