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Research Article

Weighted Hardy Operators in Complementary Morrey Spaces

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We study the weighted $p \to q$-boundedness of the multidimensional weighted Hardy-type operators H_w^α and $\mathscr{H}_w^\alpha$ with radial type weight $w = w(|x|)$, in the generalized complementary Morrey spaces ${}^{\complement}\mathcal{L}_{\{0\}}^{p,\psi}(\mathbb{R}^n)$ defined by an almost increasing function $\psi = \psi(r)$. We prove a theorem which provides conditions, in terms of some integral inequalities imposed on ψ and w, for such a boundedness. These conditions are sufficient in the general case, but we prove that they are also necessary when the function ψ and the weight w are power functions. We also prove that the spaces ${}^{\complement}\mathcal{L}_{\{0\}}^{p,\psi}(\Omega)$ over bounded domains Ω are embedded between weighted Lebesgue space L^p with the weight ψ and such a space with the weight ψ, perturbed by a logarithmic factor. Both the embeddings are sharp.

1. Introduction

Hardy operators and related Hardy inequalities are widely studied in various function spaces, and we refer to the books [1–4] and references therein. They continue to attract attention of researchers both as an interesting mathematical object and a useful tool for many purposes: see for instance the recent papers [5, 6]. Results on weighted estimations of Hardy operators in Lebesgue spaces may be found in the abovementioned books. In the papers [7–9] the weighted boundedness of the Hardy type operators was studied in Morrey spaces.

In this paper we study multi-dimensional weighted Hardy operators

$$H_w^\alpha f(x) := |x|^{\alpha-n} w(|x|) \int_{|y|<|x|} \frac{f(y)dy}{w(|y|)}, \qquad \mathscr{H}_w^\alpha f(x) := |x|^\alpha w(|x|) \int_{|y|>|x|} \frac{f(y)dy}{|y|^n w(|y|)}, \tag{1.1}$$

where $\alpha \geq 0$, in the so called *complementary Morrey spaces*. The one-dimensional case will include the versions

$$H_w^\alpha f(x) = x^{\alpha-1} w(x) \int_0^x \frac{f(t)dt}{w(t)}, \quad \mathcal{H}_w^\alpha f(x) = x^\alpha w(x) \int_x^\infty \frac{f(t)dt}{tw(t)}, \quad x > 0 \tag{1.2}$$

adjusted for the half-axis $\mathbb{R}_+^1$, so that in the sequel $\mathbb{R}^n$ with $n = 1$ may be read either as $\mathbb{R}^1$ or $\mathbb{R}_+^1$.

The classical Morrey spaces $\mathcal{L}^{p,\lambda}(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, defined by the norm

$$\|f\|_{p,\lambda} := \sup_{x\in\Omega, r>0} \left(\frac{1}{r^\lambda} \int_{\widetilde{B}(x,r)} |f(y)|^p dy \right)^{1/p}, \quad 1 \leq p < \infty, \ 0 \leq \lambda \leq n, \tag{1.3}$$

where $\widetilde{B}(x,r) = \Omega \cap B(x,r)$, are well known, in particular, because of their usage in the study of regularity properties of solutions to PDE; see for instance the books [10–12] and references therein. There are also known various generalizations of the classical Morrey spaces $\mathcal{L}^{p,\lambda}$, and we refer for instance to the surveying paper [13]. One of the direct generalizations is obtained by replacing r^λ in (1.3) by a function $\varphi(r)$, usually satisfying some monotonicity type conditions. We also denote it as $\mathcal{L}^{p,\varphi}(\Omega)$ without danger of confusion. Such spaces appeared in [14, 15] and were widely studied in [16, 17]. The spaces $\mathcal{L}_{\mathrm{loc}}^{p,\varphi}(\Omega)$, defined by the norm

$$\|f\|_{p,\varphi;\mathrm{loc}} := \sup_{r>0} \left(\frac{1}{\varphi(r)} \int_{\widetilde{B}(x_0,r)} |f(y)|^p \, dy \right)^{1/p}, \tag{1.4}$$

where $x_0 \in \Omega$, are known as generalized *local Morrey spaces*

Hardy type operators (1.1) in the spaces $\mathcal{L}^{p,\varphi}(\Omega)$, $\mathcal{L}_{\mathrm{loc}}^{p,\varphi}(\Omega)$ have been studied in [7–9].

The norm in Morrey spaces controls the smallness of the integral $\int_{B(x,r)} |f(y)|^p \, dy$ over small balls $B(x,r)$ (and also a possible growth of this integral for $r \to \infty$ in the case Ω is unbounded). There are also known spaces ${}^{\complement}\mathcal{L}_{\{x_0\}}^{p,\psi}(\Omega)$, called *complementary Morrey spaces*, with the norm controlling possible growth, as $r \to 0$, of the integral

$$\int_{\Omega\backslash B(x_0,r)} |f(y)|^p \, dy \tag{1.5}$$

over exterior of balls. Such spaces have sense in the local setting only. It is introduced in [16, 17] that the space ${}^{\complement}\mathcal{L}_{\{x_0\}}^{p,\psi}(\Omega)$ is defined as the space of all functions $f \in L_{\mathrm{loc}}^p(\Omega \setminus \{x_0\})$ with the finite norm

$$\|f\|_{{}^{\complement}\mathcal{L}_{\{x_0\}}^{p,\psi}(\Omega)} = \sup_{r>0} \psi^{1/p}(r) \|f\|_{L^p(\Omega\backslash B(x_0,r))}, \tag{1.6}$$

where admission of the multiplier $\psi(r)$ vanishing at $r = 0$ controls the growth of the norm $\|f\|_{L^p(\Omega\backslash B(x_0,r))}$. Such spaces were also studied in some later papers, and we refer for instance to [18]. We refer also to the paper [19] where variable exponent complementary spaces of such type were introduced.

During the last decades various classical operators, such as maximal, singular, and potential operators were widely investigated both in classical and generalized Morrey spaces, including the complementary Morrey spaces. The mapping properties of Hardy type operators in complementary Morrey spaces were not known. In this paper we obtain conditions for the $p \to q$-boundedness of Hardy type operators in complementary Morrey spaces. Thus we make certain contributions to the known theory of Hardy type inequalities; see, for example, the books [2–4] and references given there.

We also prove a new property for the generalized complementary Morrey spaces; see Theorem 3.1, by showing that the spaces ${}^{\complement}\mathcal{L}^{p,\varphi}_{\{0\}}(\Omega)$ over bounded domains Ω are embedded between the weighted Lebesgue space L^p with the weight φ, and such a space with the weight φ, perturbed by a logarithmic factor. In the case where φ was a power function, this was proved in [19].

The paper is organized as follows. In Section 2 we give definitions and necessary preliminaries, including conditions for radial functions to belong to complementary Morrey spaces. In Section 3 we prove the abovementioned embedding of the generalized Morrey space between Lebesgue weighted spaces. In Section 4, which plays a crucial role in the preparation of the proofs of the main results, we prove pointwise estimate for the Hardy-type constructions via the norm defining the complementary Morrey space. In Section 5 we give the final theorems on the weighted $p \to q$-boundedness of Hardy operators in complementary Morrey spaces. In the appendix we collect various properties of weights from the Bary-Stechkin class which we need when we formulate some sufficient conditions for the boundedness in terms of the Matuszewska-Orlicz indices of the functions φ and w.

2. Complementary Morrey Spaces and Their Properties

2.1. Definitions

Let Ω be an open set in $\mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$ and $\ell = \operatorname{diam}\ \Omega$, $0 < \ell \leq \infty$, $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$ and $\widetilde{B}(x,r) = B(x,r) \cap \Omega$. Let also $\varphi(r)$ be a continuous function nonnegative on $(0,\ell)$ with $\varphi(0) := \lim_{r\to 0} \varphi(r) = 0$ which may arbitrarily grow as $r \to \infty$. Let also $1 \leq p < \infty$.

Definition 2.1. Let $1 \leq p < \infty$. The complementary Morrey space ${}^{\complement}\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega)$, where $x_0 \in \Omega$, is the space of functions $f \in L^p_{\text{loc}}(\Omega \setminus \{x_0\})$ such that

$$\|f\|_{{}^{\complement}\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega)} := \sup_{r>0} \left(\varphi(r) \int_{\Omega\setminus\widetilde{B}(x_0,r)} |f(y)|^p \, dy \right)^{1/p} < \infty. \tag{2.1}$$

In the case of power factor $\varphi(r) \equiv r^\lambda$, $\lambda > 0$, we also denote

$$ {}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\Omega) := {}^{\complement}\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega)\big|_{\varphi\equiv r^\lambda} \tag{2.2}$$

without danger of confusion.

The space ${}^{\complement}\mathcal{L}^{p,\psi}_{\{x_0\}}(\Omega)$ is nontrivial in the case of any locally bounded function $\psi(r)$ with an arbitrary behaviour at infinity, because bounded functions with compact support belong then to this space.

We will also use the following notation for the modular:

$$\mathcal{M}_{p,\psi}(f;x_0,r) := \psi(r)\int_{\Omega\setminus\widetilde{B}(x_0,r)} |f(y)|^p \, dy. \tag{2.3}$$

The function ψ, defining the complementary Morrey space, will be called *definitive* function.

Remark 2.2. The function φ in (1.4), defining the Morrey spaces, in some papers is called *weight* function. We prefer to call both φ and ψ as *definitive functions*, keeping the word *weight* for its natural use in the theory of function spaces, that is, for the cases where the function f itself is controlled by a weight.

2.2. On Belongness of Radial Functions to the Space ${}^{\complement}\mathcal{L}^{p,\psi}_{\{x_0\}}(\Omega)$

Lemma 2.3. *For a nonnegative radial type functions $u = u(|x-x_0|)$, $x_0 \in \Omega$, the condition*

$$\sup_{0<r<\ell}\psi(r)\int_r^{\ell} u(t)^p t^{n-1}\, dt < \infty \tag{2.4}$$

is sufficient to belong to ${}^{\complement}\mathcal{L}^{p,\psi}_{\{x_0\}}(\Omega)$. (It is also necessary, when $\Omega = \mathbb{R}^n$ or Ω is a ball centered at x_0.)

The proof of Lemma 2.3 is obvious. By means of this lemma we easily obtain the following corollary.

Corollary 2.4. *Let Ω be bounded. A power function $|x-x_0|^{\gamma}$ with $\gamma \neq -n/p$ belongs to the space ${}^{\complement}\mathcal{L}^{p,\psi}_{\{x_0\}}(\Omega)$ if and only if*

$$\sup_{r>0} r^{\min\{n+\gamma p,0\}}\psi(r) < \infty. \tag{2.5}$$

In the case $n+\gamma p = 0$, the same holds with $r^{\min\{n+\gamma p,0\}}$ replaced by $|\ln\ r|$. When $\Omega = \mathbb{R}^n$, the necessary and sufficient conditions are $n+\gamma p < 0$ and $\sup_{r>0} r^{n+\gamma p}\psi(r) < \infty$.

As is known, there may be given sufficient numerical inequalities for the validity of the integral condition in (2.4) in terms of Matuszewska-Orlicz indices $m(u)$ and $M(u)$ of the function u. We give below such sufficient inequalities, but in order not to interrupt the main body of the paper by notions connected with such indices and related Bary-Stechkin function classes $\mathbb{Z}_{\alpha,\beta}$, we put these notions in the appendix.

Let now u be a nonnegative function in Bary-Stechkin class, namely,

$$u \in \mathbb{Z}_{-n/p}([0,\ell]) \quad \text{if } \ell < \infty, \qquad u \in \mathbb{Z}_{-n/p,-n/p}\left(\mathbb{R}^1_+\right) \quad \text{if } \ell = \infty. \tag{2.6}$$

Then $\int_r^\ell u^p(t)t^{n-1}\,dt \sim r^n u^p(r)$, $0 < r < \ell$, so that the condition $\sup_{0<r<\ell} r^n \varphi(r) u^p(r) < \infty$ is sufficient for $u = u(|x-x_0|)$ to be in ${}^{\complement}\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega)$. Note also that (2.6) is equivalent to the inequalities $pM(u)+n<0$, $pM_\infty(u)+n<0$ in terms of the indices, where the second inequality is to be used only in the case $\ell = \infty$; see (A.16).

3. Complementary Morrey Spaces Are Closely Embedded between Weighted Lebesgue Spaces

The complementary Morrey spaces are close to a certain weighted Lebesgue space as stated in the following theorem, which provides sharp embeddings of independent interest. The notation for the weighted space is taken in the form $L^p(\Omega,\varrho) := \{f : \int_\Omega \varrho(x)|f(x)|^p dx < \infty\}$. The class $W_0([0,\ell])$, used in this theorem, is defined in the appendix; see Definitions A.1-A.2 therein. In the case where $\varphi(r)$ is a power function, Theorem 3.1 was proved in [19].

Theorem 3.1. *Let Ω be a bounded open set, $1 \le p < \infty$, $\varphi \in W_0([0,\ell])$, $\ell = \operatorname{diam}\Omega$, and let φ be absolutely continuous and*

$$P := \sup_{0<t<\ell} \frac{t\left|\varphi'(t)\right|}{\varphi(t)} < \infty. \tag{3.1}$$

Then

$$L^p(\Omega, \varphi(|y-x_0|)) \hookrightarrow {}^{\complement}\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega) \hookrightarrow \bigcap_{\varepsilon>0} L^p(\Omega, \varphi_\varepsilon(|y-x_0|)), \tag{3.2}$$

where $\varphi_\varepsilon(t) = \varphi(t)/(\ln(A/t))^{1+\varepsilon}$, $A > \ell$. The left-hand side embedding in (3.2) *is strict, when φ satisfies the condition*

$$\int_r^\ell \frac{dt}{t\varphi(t)} dt \le \frac{C}{\varphi(r)} \tag{3.3}$$

(in particular, if $\varphi(r) = r^\lambda$, $\lambda > 0$); that is, there exists a function $f_0 = f_0(x)$ such that

$$f_0 \in {}^{\complement}\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega), \quad \text{but } f_0 \notin L^p(\Omega, \varphi(|y-x_0|)). \tag{3.4}$$

The right-hand side embedding in (3.2) *is strict, when φ satisfies the doubling condition $\varphi(2r) \le C\varphi(r)$, (in particular, if $\varphi(r) = r^\lambda$, $\lambda \ge 0$), there exists a function $g_0 = g_0(x)$ such that*

$$g_0 \in \bigcap_{\varepsilon>0} L^p(\Omega, \varphi_\varepsilon(|x-x_0|)), \quad \text{but } g_0 \notin {}^{\complement}\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega). \tag{3.5}$$

Proof. Without loss of generality, we may take $x_0 = 0$ for simplicity, supposing that $0 \in \Omega$.

1^0. *The left-hand side embedding*: since the function φ is almost increasing, we have

$$\begin{aligned}\left(\int_\Omega \varphi(|y|) f(y)|^p\, dy\right)^{1/p} &\geq \left(\int_{\Omega\setminus\widetilde{B}(0,r)} \varphi(|y|) f(y)|^p\, dy\right)^{1/p} \\ &\geq C\left(\varphi(r)\int_{\Omega\setminus\widetilde{B}(0,r)} |f(y)|^p\, dy\right)^{1/p}\end{aligned} \tag{3.6}$$

so that

$$\|f\|_{L^p(\Omega,\varphi(|y|))} \geq C\|f\|_{{}^{\complement}\mathcal{L}^{p,\varphi}_{\{0\}}(\Omega)}. \tag{3.7}$$

2^0. The right-hand side embedding: we have

$$\int_{\widetilde{B}(0,r)} |f(y)|^p \varphi_\varepsilon(|y|)dy = \int_{\widetilde{B}(0,r)} |f(y)|^p\left(\int_0^{|y|} \frac{d}{dt}\varphi_\varepsilon(t)dt\right)dy, \tag{3.8}$$

where

$$\varphi_\varepsilon'(t) = \frac{\varphi(t)}{t}\left[\frac{t\varphi'(t)}{\varphi(t)} + \frac{1+\varepsilon}{\ln(A/t)}\right]\frac{1}{(\ln(A/t))^{1+\varepsilon}}, \tag{3.9}$$

so that

$$|\varphi_\varepsilon'(t)| \leq \frac{C_\varepsilon\varphi(t)}{t(\ln(A/t))^{1+\varepsilon}}, \qquad C_\varepsilon = P + \frac{1+\varepsilon}{\ln(A/\ell)}. \tag{3.10}$$

Therefore,

$$\begin{aligned}\int_{\widetilde{B}(x_0,r)} |f(y)|^p \varphi_\varepsilon(|y|)dy &\leq \int_0^t |\varphi_\varepsilon'(t)| \left(\int_{\{y\in\Omega: t<|x_0-y|<r\}} |f(y)|^p dy\right)dt \\ &\leq \int_0^\ell |\varphi_\varepsilon'(t)| \cdot \|f\|^p_{L^p(\Omega\setminus\widetilde{B}(x_0,t))} ds \\ &\leq \|f\|^p_{{}^{\complement}\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega)} \int_0^\ell \frac{|\varphi_\varepsilon'(t)|}{\varphi(t)}\, dt,\end{aligned} \tag{3.11}$$

where the last integral converges when $\varepsilon > 0$ since $|\varphi_\varepsilon'(t)|/\varphi(t) \leq C_\varepsilon/(t(\ln(A/t))^{1+\varepsilon})$.

3^0. *The strictness of the embeddings*: the corresponding counterexamples are

$$f_0(x) = \frac{1}{\varphi^{1/p}(|x|)|x|^{n/p}}, \qquad g_0(x) = \frac{\ln(\ln(B/|x|))}{|x|^{n/p}\varphi^{1/p}(|x|)}, \quad B > \ell e^e. \tag{3.12}$$

Calculations for the function f_0, which is obviously not in $L^p(\Omega, \psi(|y|)$, are easy. Indeed,

$$\|f_0\|^p_{^{\complement}\mathcal{L}^{p,\psi}_{\{0\}}(\Omega)} \leq \left|\mathbb{S}^{n-1}\right| \psi(r) \int_r^{\ell} \frac{dt}{t\psi(t)}, \tag{3.13}$$

where the right-hand side is bounded by (3.3).

In the case of the function g_0 we have

$$\|g_0\|^p_{L^p(\Omega,\psi_\varepsilon)} = \int_\Omega \frac{\ln^p(\ln(B/|x|))}{|x|^n(\ln(A/|x|))^{1+\varepsilon}} dx \leq \left|\mathbb{S}^{n-1}\right| \int_0^{\ell} \frac{\ln^p(\ln(B/t))}{t(\ln(A/t))^{1+\varepsilon}} dt < \infty \tag{3.14}$$

for every $\varepsilon > 0$. However, for small $r \in (0, \delta/2)$, where $\delta = \operatorname{dist}(0, \partial\Omega)$, we obtain

$$\begin{aligned}
\psi(r) \int_{x\in\Omega:\ |x|>r} g_0^p(|x|) dx &\geq \psi(r) \int_{x\in\Omega:\ r<|x|<\delta} \frac{\ln^p(\ln(B/|x|))dx}{|x|^n \psi(|x|)} \\
&= \left|\mathbb{S}^{n-1}\right| \psi(r) \int_r^{\delta} \frac{\ln^p(\ln(B/t))dt}{t\psi(t)} \\
&\geq \left|\mathbb{S}^{n-1}\right| \psi(r) \int_r^{2r} \frac{\ln^p(\ln(B/t))dt}{t\psi(t)}.
\end{aligned} \tag{3.15}$$

Taking into account that $\psi(t)$ is almost increasing and satisfies the doubling condition, we get

$$\psi(r) \int_r^{2r} \frac{\ln^p(\ln(B/t))dt}{t\psi(t)} \geq C \frac{\psi(r)}{\psi(2r)} \ln^p\left(\ln \frac{B}{2r}\right) \int_r^{2r} \frac{dt}{t} \geq C \ln^p\left(\ln \frac{B}{r}\right) \longrightarrow \infty \quad \text{as } r \longrightarrow 0, \tag{3.16}$$

which completes the proof of the theorem. □

Remark 3.2. Under the assumption (3.3), the upper embedding in (3.2) does not hold with $\varepsilon = 0$. The corresponding counterexample is $f(x) = 1/(|x - x_0|^n \psi(|x - x_0|))$ which is in $^{\complement}\mathcal{L}^{p,\psi}_{\{x_0\}}(\Omega)$, but $f \notin L^p(\Omega, \psi_\varepsilon(|y - x_0|))|_{\varepsilon=0}$.

4. Weighted Estimates of Functions in Complementary Morrey Spaces

In the following lemmas we give a pointwise estimate of the "Hardy-type" constructions in terms of the modular $\mathfrak{M}_{p,\psi}(f; x_0, r)$ with $x_0 = 0$. These estimates are of independent interest and also crucial for our study of the Hardy operators in the complementary Morrey space.

Lemma 4.1. *Let $1 \leq p < \infty$, $0 < s \leq p$, let $v, \varphi^{s/p} v \in \overline{W}([0,\infty])$, and $v(2t) \leq cv(t)$. Then*

$$\int_{|z|<r} \frac{|f(z)|^s}{v(|z|)}\, dz \leq C \int_0^r \frac{V_s(t)}{t}\, dt \cdot \mathcal{M}^{s/p}_{p,\varphi}(f;0,r), \quad r > 0, \tag{4.1}$$

for $f \in L^p_{\mathrm{loc}}(\Omega \setminus \{0\})$, where $C > 0$ does not depend on f and $r \in (0,\infty)$ and

$$V_s(t) = \frac{t^{n(1-(s/p))}}{v(t)\varphi^{s/p}(t)}. \tag{4.2}$$

Proof. We use the dyadic decomposition as follows:

$$\int_{|z|<r} \frac{|f(z)|^s}{v(|z|)}\, dz = \sum_{k=0}^{\infty} \int_{B_k(r)} \frac{|f(z)|^s}{v(|z|)}\, dz, \tag{4.3}$$

where $B_k(r) = \{z : 2^{-k-1}r < |z| < 2^{-k}r\}$. Since there exists a β such that $t^{\beta}v(t)$ is almost increasing, we observe that

$$\frac{1}{v(|z|)} \leq \frac{C}{v(2^{-k-1}r)} \tag{4.4}$$

on $B_k(r)$. Applying this in (4.3) and making use of the Hölder inequality with the exponent $p/s \geq 1$, we obtain

$$\int_{|z|<r} \frac{|f(z)|^s}{v(|z|)} dz \leq C \sum_{k=0}^{\infty} \frac{(2^{-k-1}r)^{n(1-(s/p))}}{v(2^{-k-1}r)} \left(\int_{B_k(r)} |f(z)|^p\, dz \right)^{s/p}. \tag{4.5}$$

Hence

$$\int_{|z|<r} \frac{|f(z)|^s}{v(|z|)} dz \leq C \sum_{k=0}^{\infty} \left(2^{-k-1}r\right)^{n(1-(s/p))} \frac{\mathcal{M}^{s/p}_{p,\varphi}(f;0,2^{-k-1}r)}{v(2^{-k-1}r)\varphi^{s/p}(2^{-k-1}r)}. \tag{4.6}$$

On the other hand we have

$$\int_0^r \frac{V_s(t)}{t} \mathcal{M}^{s/p}_{p,\varphi}(f;0,t) dt = \sum_{k=0}^{\infty} \int_{2^{-k-1}r}^{2^{-k}r} t^{n(1-(s/p))-1} \frac{\mathcal{M}^{s/p}_{p,\varphi}(f;0,t)}{v(t)\varphi^{S/p}(t)} dt. \tag{4.7}$$

The function $1/(\psi(t))\mathcal{M}_{p,\psi}(f;0,t)$ is decreasing, and the function $1/(v(t))$ is almost decreasing after multiplication by some power functions. Therefore,

$$\begin{aligned}\int_0^r \frac{V_s(t)}{t}\mathcal{M}_{p,\psi}^{s/p}(f;0,t)dt &\geq C\sum_{k=0}^{\infty}\left(2^{-k}r\right)^{n(1-(s/p))}\frac{\mathcal{M}_{p,\psi}^{s/p}(f;0,2^{-k}r)}{v(2^{-k}r)\psi^{S/p}(2^{-k}r)} \\ &\geq C\sum_{k=1}^{\infty}\left(2^{-k}r\right)^{n(1-(s/p))}\frac{\mathcal{M}_{p,\psi}^{s/p}(f;0,2^{-k}r)}{v(2^{-k}r)\psi^{s/p}(2^{-k}r)} \\ &= C_1\sum_{k=0}^{\infty}\left(2^{-k-1}r\right)^{n(1-(s/p))}\frac{\mathcal{M}_{p,\psi}^{s/p}(f;0,2^{-k-1}r)}{v(2^{-k-1}r)\psi^{s/p}(2^{-k-1}r)}.\end{aligned} \tag{4.8}$$

Then (4.1) follows from (4.6) and (4.8). □

Lemma 4.2. *Let $1 \leq p < \infty$, $0 \leq s \leq p$, $v \in \overline{W}(\mathbb{R}_+^1)$ and $v(2t) \leq Cv(t)$, $\psi(2t) \leq C\psi(t)$. Let also*

$$V^s(t) := \frac{t^{n(1-(s/p))}v(t)}{\psi^{s/p}(t)}. \tag{4.9}$$

Then

$$\int_{|z|>r} v(|z|)\left|f(z)\right|^s dz \leq C\int_r^{\infty}\frac{V^s(t)}{t}\mathcal{M}_{p,\psi}^{s/p}(f;0,t)dt, \tag{4.10}$$

where $C > 0$ does not depend on $r > 0$ and f.

Proof. We use the corresponding dyadic decomposition:

$$\int_{|z|>r} v(t)\left|f(z)\right|^s \, dz = \sum_{k=0}^{\infty}\int_{B^k(r)} v(z)\left|f(z)\right|^s dz, \tag{4.11}$$

where $B^k(r) = \{z : 2^k r < |z| < 2^{k+1}r\}$. Since there exists a $\beta \in \mathbb{R}^1$ such that $t^{\beta}v(t)$ is almost increasing, we obtain

$$\sum_{k=0}^{\infty}\int_{B^k(r)} v(|z|)\left|f(z)\right|^s dz \leq C\sum_{k=0}^{\infty} v\left(2^{k+1}r\right)\int_{B^k(r)}\left|f(z)\right|^s dz, \tag{4.12}$$

where C may depend on β but does not depend on r and f. Applying the Hölder inequality with the exponent p/s, we get

$$\begin{aligned}\int_{|z|>r} v(|z|)\left|f(z)\right|^s dz &\leq C\sum_{k=0}^{\infty} v\left(2^{k+1}r\right)\left(2^k r\right)^{n(1-(s/p))}\left(\int_{B^k(r)}\left|f(z)\right|^p dz\right)^{s/p} \\ &\leq C\sum_{k=0}^{\infty} v\left(2^{k+1}r\right)\left(2^k r\right)^{n(1-(s/p))}\psi^{-s/p}\left(2^k r\right)\mathcal{M}_{p,\psi}^{s/p}\left(f;0,2^k r\right).\end{aligned} \tag{4.13}$$

On the other hand, the integral on the right-hand side of (4.10) can be estimated as follows:

$$\begin{aligned}\int_r^\infty \frac{V^s(t)}{t}\mathcal{M}_{p,\varphi}^{s/p}(f;0,t)\,dt &= \sum_{k=0}^{\infty}\int_{2^k r}^{2^{k+1}r} t^{n-1-(ns/p)}\left[\frac{\mathcal{M}_{p,\varphi}(f;0,t)}{\varphi(t)}\right]^{s/p} v(t)\,dt \\ &\geq C\sum_{k=0}^{\infty} v\left(2^k r\right)\left[\frac{\mathcal{M}_{p,\varphi}(f;0,2^{k+1}r)}{\varphi(2^{k+1}r)}\right]^{s/p}\left(2^k r\right)^{n(1-s/p)} \\ &\geq C\sum_{k=0}^{\infty} v\left(2^{k+1}r\right)\varphi^{-1/p}\left(2^k r\right)\left(2^k r\right)^{n(1-s/p)}\mathcal{M}_{p,\varphi}^{s/p}\left(f;0,2^{k+1}r\right),\end{aligned} \tag{4.14}$$

which completes the proof. □

Corollary 4.3. *Let $1 \le p < \infty$, let $w, w\varphi^{1/p} \in \overline{W}([0,\ell]), w(2t) \le cw(t)$, $0 < \ell \le \infty$ and $0 \in \Omega$. Let also*

$$V(r) := \frac{1}{w^p(r)\varphi(r)} \in \mathbb{Z}_0. \tag{4.15}$$

Then

$$\mathcal{M}_{p,\varphi}\left(\frac{f}{w};0,r\right) \le \frac{C}{w^p(r)}\mathcal{M}_{p,\varphi}(f;0,r), \quad 0 < r < \ell. \tag{4.16}$$

5. Weighted Hardy Operators in Complementary Morrey Spaces

5.1. Pointwise Estimations

The proof of our main result of this Section given in Theorems 5.3 and 5.6 is prepared by the following Theorems 5.1 and 5.2 on the pointwise estimates of the Hardy-type operators.

Theorem 5.1. *Let $1 \le p \le \infty$, let $w, w\varphi^{1/p} \in \overline{W}$, and $w(2t) \le Cw(t)$. The condition*

$$\int_0^\varepsilon \frac{V(t)}{t}\,dt < \infty \quad \text{with } \varepsilon > 0, \quad \text{where } V(t) = \frac{t^{n/p'}}{w(t)\varphi^{1/p}(t)} \tag{5.1}$$

is sufficient for the Hardy operator H_w^α to be defined on the space ${}^{\complement}\mathcal{L}_{\{0\}}^{p,\varphi}(\mathbb{R}^n)$. Under this condition, the pointwise estimate

$$\left|H_w^\alpha f(x)\right| \le C|x|^{\alpha-n}w(|x|)\int_0^{|x|}\frac{V(t)}{t}\,dt\|f\|_{{}^{\complement}\mathcal{L}_{\{0\}}^{p,\varphi}} \tag{5.2}$$

holds.

Proof. Apply Lemma 4.1 with $s = 1$. □

Theorem 5.2. *Let $1 \le p \le \infty$ and $1/w \in \overline{W}$, or $w \in \overline{W}$ and $w(2t) \le Cw(t)$. The condition*

$$\int_{\varepsilon}^{\infty} \frac{\mho(t)}{t}\, dt < \infty, \quad \text{with } \varepsilon > 0, \quad \text{where } \mho(t) = \frac{1}{w(t)\psi^{1/p}(t)t^{n/p}} \tag{5.3}$$

is sufficient for the Hardy operator $\mathscr{H}_w^\alpha$ to be defined on the space ${}^{\complement}\mathcal{L}_{\{x_0\}}^{p,\psi}(\mathbb{R}^n)$, and in this case the following pointwise estimate:

$$\left|\mathscr{H}_w^\alpha f(x)\right| \le C|x|^\alpha w(|x|) \int_{|x|}^{\infty} \frac{\mho(t)}{t} dt \|f\|_{{}^{\complement}\mathcal{L}_{\{x_0\}}^{p,\psi}} \tag{5.4}$$

holds.

Proof. Apply Lemma 4.2 with $s = 1$. □

5.2. *Weighted* $p \to q$ *Boundedness of Hardy Operators in Complementary Morrey Spaces*

5.2.1. *The Case of the Operator H_w^α*

The $L^p \to L^q$-boundedness of the multidimensional Hardy operators within the frameworks of Lebesgue spaces (the case $\varphi \equiv 1$ in (1.4)) with $1 < p < \infty$ and $0 < q < \infty$ is well known: see for instance, [4, page 54]. For Morrey spaces, both local and global, the boundedness of Hardy operators was studied in [7, 8]. We call attention of the reader to the fact that, in contrast to the case of Lebesgue spaces, Hardy-type inequalities in both usual and complementary Morrey spaces different from Lebesgue spaces (i.e., in the case $\varphi(0) = 0$ or $\psi(0) = 0$) admit the value $p = 1$.

Theorem 5.3. *Let $1 \le p \le \infty, 1 \le q \le \infty$ and let*

$$w, \psi^{1/p} w \in \overline{W}\left(\mathbb{R}_+^1\right), \qquad w(2t) \le cw(t). \tag{5.5}$$

The operator H_w^α is bounded from ${}^{\complement}\mathcal{L}_{\{0\}}^{p,\psi}(\mathbb{R}^n)$ to ${}^{\complement}\mathcal{L}_{\{0\}}^{q,\psi}(\mathbb{R}^n)$ if $\sup_{r>0} \mathbb{W}(r) < \infty$, where

$$\mathbb{W}(r) := \psi(r) \int_r^\infty \varrho^{n-1} \left(\frac{\varrho^{(\alpha-(n/p))}}{\psi^{1/p}(\varrho) V(\varrho)} \int_0^\varrho \frac{V(t)}{t} dt \right)^q d\varrho, \tag{5.6}$$

and $V(t)$ is the same as in (5.1)*. Under this condition,*

$$\left\|H_w^\alpha f\right\|_{{}^{\complement}\mathcal{L}_{\{0\}}^{q,\psi}} \le C \sup_{r>0} \mathbb{W}^{1/q}(r) \|f\|_{{}^{\complement}\mathcal{L}_{\{0\}}^{p,\psi}}. \tag{5.7}$$

Proof. From the estimate (5.2) of Theorem 5.1 we have

$$\left|H_w^\alpha f(x)\right| \leq C \frac{|x|^{\alpha-(n/p)}}{\varphi^{q/p}(|x|)} \left(\frac{1}{V(|x|)} \int_0^{|x|} \frac{V(t)}{t}\, dt\right) \|f\|_{{}^{\complement}\mathcal{L}^{q,\varphi}_{\{0\}}}. \tag{5.8}$$

Hence, calculating $\|H_w^\alpha f\|_{{}^{\complement}\mathcal{L}^{p,\varphi}_{\{0\}}}$, passing to polar coordinates, we obtain (5.6). □

Remark 5.4. Note that

$$\frac{1}{V(\varrho)} \int_0^{\varrho} \frac{V(t)}{t} dt \simeq 1 \tag{5.9}$$

in (5.6) if we suppose that $V \in \mathbb{Z}^0$.

The following corollary gives sufficient conditions for the boundedness of the operator H_w^α in terms of the Matuszewska-Orlicz indices of the function φ and the weight w (we refer to the appendix for these notions).

Corollary 5.5. *Let* $1 \leq p \leq q < \infty$ *(with* $q = p$ *admitted in the case* $\alpha = 0$*) and the conditions* (5.5) *be satisfied. Suppose also that*

$$\varphi(r) \geq C r^{(\alpha pq/q-p)-n} \tag{5.10}$$

in the case $\alpha \neq 0$. *The operator* H_w^α *is bounded from* ${}^{\complement}\mathcal{L}^{p,\varphi}_{\{0\}}(\mathbb{R}^n)$ *to* ${}^{\complement}\mathcal{L}^{q,\varphi}_{\{0\}}(\mathbb{R}^n)$ *if*

$$\min\{m(V), m_\infty(V)\} > 0, \qquad \min\{m(\varphi), m_\infty(\varphi)\} > 0. \tag{5.11}$$

The condition $\min\{m(V), m_\infty(V)\} > 0$, *is guaranteed by the inequalities*

$$M(w) < \frac{n}{p'} - M(\varphi), \qquad M_\infty(w) < \frac{n}{p'} - M_\infty(\varphi). \tag{5.12}$$

In the power case $\varphi(r) = r^\lambda$, $\lambda > 0$, *and* $w(r) = r^\mu$, *the conditions* (5.10)-(5.11) *reduce to*

$$1 \leq p < \frac{n+\lambda}{\alpha}, \qquad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n+\lambda}, \qquad \mu < \frac{n}{p'} - \frac{\lambda}{p}; \tag{5.13}$$

conditions (5.13) *are also necessary for the operator* H_w^α *to be bounded from* ${}^{\complement}\mathcal{L}^{p,\varphi}_{\{0\}}(\mathbb{R}^n)$ *to* ${}^{\complement}\mathcal{L}^{q,\varphi}_{\{0\}}(\mathbb{R}^n)$.

Proof. We have to check that the condition $\sup_{r>0} \mathbb{W}(r) < \infty$ of Theorem 5.3 holds under the assumptions (5.10)-(5.11). From the inequality $\min\{m(V), m_\infty(V)\} > 0$ it follows that

$$\mathbb{W}(r) \leq C\varphi(r) \int_r^\infty \varrho^{n-1} \frac{\varrho^{q(\alpha-(n/p))}}{\varphi^{q/p}(\varrho)}\, d\varrho. \tag{5.14}$$

We represent $\psi^{q/p}(\varrho)$ as $\psi^{q/p}(\varrho) = \psi^{(q/p)-1}(\varrho)\psi(\varrho)$ and by (5.10) obtain

$$\mathbb{W}(r) \leq C\psi(r)\int_r^\infty \frac{d\varrho}{\varrho\psi(\varrho)}, \tag{5.15}$$

which is bounded in view of the second assumption in (5.11), by the property (A.4).

The sufficiency of the conditions (5.13) in the case of power functions is then obvious since $m(\psi) = m_\infty(\psi) = \lambda$ and $m(w) = m_\infty(w) = \mu$ in this case.

Let us show the necessity of these conditions. From the boundedness $\|H_w^\alpha\|_{{}^{\complement}\mathcal{L}^{q,\psi}_{\{0\}}(\mathbb{R}^n)} \leq C\|f\|_{{}^{\complement}\mathcal{L}^{p,\psi}_{\{0\}}(\mathbb{R}^n)}$ with $\psi(r) = r^\lambda$, $\lambda > 0$, and $w(r) = r^\mu$, by standard homogeneity arguments with the use of the dilation operator $\Pi_\delta f(x) := f(\delta x)$ it is easily derived that the conditions $1 \leq p < (n+\lambda)/\alpha$, $1/q = (1/p) - (\alpha/(n+\lambda))$ necessarily hold, via the relation

$$\|\Pi_\delta f\|_{{}^{\complement}\mathcal{L}^{p,\psi}_{\{0\}}(\mathbb{R}^n)} = \delta^{-(n+\lambda)/p}\|f\|_{{}^{\complement}\mathcal{L}^{p,\psi}_{\{0\}}(\mathbb{R}^n)}, \qquad H_w^\alpha \Pi_\delta f = \delta^{-\alpha}\Pi_\delta H_w^\alpha f. \tag{5.16}$$

(We take into account that we excluded $q = \infty$ in this theorem.)

The necessity of the remaining condition $\mu < (n/p') - (\lambda/p)$ in (5.13) follows from the fact that $|y|^{-(n+\lambda)/p} \in {}^{\complement}\mathcal{L}^{p,\psi}_{\{0\}}(\mathbb{R}^n)$ by Corollary 2.4, so that this condition is necessary for the operator H_w^α with $w = r^\mu$ to be defined on the space ${}^{\complement}\mathcal{L}^{p,\psi}_{\{0\}}(\mathbb{R}^n)$. □

5.2.2. The Case of the Operator $\mathcal{H}_w^\alpha$

Let

$$\mathcal{W}(r) := \psi(r)\int_r^\infty w^q(\varrho)\varrho^{q\alpha+n-1}\left(\int_\varrho^\infty \frac{\mathcal{V}(t)}{t}dt\right)^q d\varrho, \tag{5.17}$$

where $\mathcal{V}$ is the same as in (5.3).

Theorem 5.6. *Let $1 \leq p < \infty$, $1 \leq q < \infty$ and*

$$w\psi^{1/p} \in \underline{W}\left(\mathbb{R}^1_+\right) \quad or \quad w \in \overline{W}\left(\mathbb{R}^1_+\right), \qquad w(2t) \leq Cw(t). \tag{5.18}$$

The operator $\mathcal{H}_w^\alpha$ is bounded from ${}^{\complement}\mathcal{L}^{p,\psi}_{\{0\}}(\mathbb{R}^n)$ to ${}^{\complement}\mathcal{L}^{q,\psi}_{\{0\}}(\mathbb{R}^n)$ if

$$\sup_{r>0}\mathcal{W}(r) < \infty, \tag{5.19}$$

and then

$$\|\mathcal{H}_w^\alpha f\|_{{}^{\complement}\mathcal{L}^{q,\psi}_{\{0\}}} \leq C\sup_{r>0}\mathcal{W}^{1/q}(r)\|f\|_{{}^{\complement}\mathcal{L}^{p,\psi}_{\{0\}}}. \tag{5.20}$$

Proof. From the estimate (5.4) we obtain

$$\mathcal{M}_{q,\psi}\left(\mathcal{H}_w^{\alpha} f;0,r\right) \le C\psi(r)\int_r^{\infty} \varrho^{n-1+q(\alpha-(n/p))}\psi^{-q/p}(\varrho)\left(\frac{1}{\mathcal{V}(\varrho)}\int_{\varrho}^{\infty}\frac{\mathcal{V}(t)}{t}dt\right)^q d\varrho \|f\|^q_{{}^{\complement}\mathcal{L}^{p,\psi}_{\{0\}}}, \tag{5.21}$$

from which (5.20) follows. □

As above, we provide also sufficient conditions for the boundedness of the operator $\mathcal{H}_w^{\beta}$ in terms of the Matuszewska-Orlicz indices.

Corollary 5.7. *Let* $1 \le p \le q < \infty$ *(with* $q = p$ *admitted in the case* $\alpha = 0$ *and* w *and* ψ *satisfy the conditions* (5.18)*. The operator* $\mathcal{H}_w^{\alpha}$ *is bounded from* ${}^{\complement}\mathcal{L}^{p,\psi}_{\{0\}}(\mathbb{R}^n)$ *to* ${}^{\complement}\mathcal{L}^{q,\psi}_{\{0\}}(\mathbb{R}^n)$ *if*

$$\max\{M(\mathcal{V}), M_{\infty}(\mathcal{V})\} < 0, \qquad \min\{m(\psi), m_{\infty}(\psi)\} > 0, \tag{5.22}$$

and the condition (5.10) *holds; the assumption* $\max\{M(\mathcal{V}), M_{\infty}(\mathcal{V})\} < 0$ *is guaranteed by the conditions*

$$m(w) > -\frac{m(\psi)+n}{p}, \qquad m_{\infty}(w) > -\frac{m_{\infty}(\psi)+n}{p}. \tag{5.23}$$

In the case $\psi(r) = r^{\lambda}$, $\lambda > 0$, *and* $w(r) = r^{\mu}$, *the conditions* (5.22) *and* (5.10) *reduce to*

$$1 \le p < \frac{n+\lambda}{\alpha}, \qquad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n+\lambda}, \qquad \mu > \frac{n+\lambda}{p}; \tag{5.24}$$

conditions (5.24) *are also necessary for the operator* $\mathcal{H}_w^{\alpha}$ *to be bounded from* ${}^{\complement}\mathcal{L}^{p,\psi}_{\{0\}}(\mathbb{R}^n)$ *to* ${}^{\complement}\mathcal{L}^{q,\psi}_{\{0\}}(\mathbb{R}^n)$.

Proof. We have to find, in terms of the Matuszewska-Orlicz indices, conditions sufficient for the validity of (5.19). For the latter we have

$$\begin{aligned}\mathcal{W}(r) &\le C\psi(r)\int_r^{\infty} \varrho^{n-1+q(\alpha-(n/p))}\psi^{-q/p}(\varrho)\left(\frac{1}{\mathcal{V}(\varrho)}\int_{\varrho}^{\infty}\frac{\mathcal{V}(t)}{t}\,dt\right)^q d\rho \\ &\le C\psi(r)\int_r^{\infty} \varrho^{n-1+q(\alpha-(n/p))}\psi^{-q/p}(\varrho)d\rho\end{aligned} \tag{5.25}$$

by the assumption $\max\{M(\mathcal{V}), M_{\infty}(\mathcal{V})\} < 0$. With $\psi^{q/p}(\rho) = \psi^{(q/p)-1}(\rho)\psi(\rho)$ and the condition (5.10) we arrive at $\mathcal{W}(r) \le C\psi(r)\int_r^{\infty} d\varrho/\varrho\psi(\varrho)$, where the boundedness of the right-hand side is guaranteed by the condition $\min\{m(\psi), m_{\infty}(\psi)\} > 0$.

The proof for the case of power functions is similar to that in Corollary 5.5. □

Appendix

A. Zygmund-Bary-Stechkin (ZBS) Classes and Matuszewska-Orlicz (MO) Type Indices

In the sequel, a nonnegative function f on $[0,\ell]$, $0<\ell\leq\infty$, is called almost increasing (almost decreasing) if there exists a constant C (≥ 1) such that $f(x)\leq Cf(y)$ for all $x\leq y$ ($x\geq y$, resp.). Equivalently, a function f is almost increasing (almost decreasing) if it is equivalent to an increasing (decreasing, resp.) function g, that is, $c_1 f(x)\leq g(x)\leq c_2 f(x)$, $c_1>0$, $c_2>0$.

Definition A.1. Let $0<\ell<\infty$:

(1) by $W=W([0,\ell])$ we denote the class of continuous and positive functions φ on $(0,\ell]$ such that there exists finite or infinite limit $\lim_{x\to 0}\varphi(x)$;

(2) by $W_0=W_0([0,\ell])$ we denote the class of almost increasing functions $\varphi\in W$ on $(0,\ell)$;

(3) by $\overline{W}=\overline{W}([0,\ell])$ we denote the class of functions $\varphi\in W$ such that $x^a\varphi(x)\in W_0$ for some $a=a(\varphi)\in\mathbb{R}^1$;

(4) by $\underline{W}=\underline{W}([0,\ell])$ we denote the class of functions $\varphi\in W$ such that $\varphi(t)/t^b$ is almost decreasing for some $b\in\mathbb{R}^1$.

Definition A.2. Let $0<\ell<\infty$:

(1) by $W_\infty=W_\infty([\ell,\infty])$ we denote the class of functions φ which are continuous and positive and almost increasing on $[\ell,\infty)$ and which have the finite or infinite limit $\lim_{x\to\infty}\varphi(x)$,

(2) by $\overline{W}_\infty=\overline{W}_\infty([\ell,\infty))$ we denote the class of functions $\varphi\in W_\infty$ such $x^a\varphi(x)\in W_\infty$ for some $a=a(\varphi)\in\mathbb{R}^1$.

By $\overline{W}(\mathbb{R}^1_+)$ we denote the set of functions on $\mathbb{R}^1_+$ whose restrictions onto $(0,1)$ are in $\overline{W}([0,1])$ and restrictions onto $[1,\infty)$ are in $\overline{W}_\infty([1,\infty))$. Similarly, the set $\underline{W}(\mathbb{R}^1_+)$ is defined.

A.1. ZBS Classes and MO Indices of Weights at the Origin

In this subsection we assume that $\ell<\infty$.

We say that a function φ belongs to a Zygmund class $\mathbb{Z}^\beta$, $\beta\in\mathbb{R}^1$, if $\varphi\in\overline{W}([0,\ell])$ and $\int_0^x(\varphi(t)/t^{1+\beta})dt\leq c(\varphi(x)/x^\beta)$, $x\in(0,\ell)$, and to a Zygmund class $\mathbb{Z}_\gamma$, $\gamma\in\mathbb{R}^1$, if $\varphi\in\underline{W}([0,\ell])$ and $\int_x^\ell(\varphi(t)/t^{1+\gamma})dt\leq c(\varphi(x)/x^\gamma)$, $x\in(0,\ell)$. We also denote

$$\Phi^\beta_\gamma:=\mathbb{Z}^\beta\bigcap\mathbb{Z}_\gamma, \tag{A.1}$$

the latter class being also known as Bary-Stechkin-Zygmund class [20].

For a function $\varphi \in \overline{W}$, the numbers

$$m(\varphi) = \sup_{0<x<1} \frac{\ln(\limsup_{h\to 0}(\varphi(hx)/\varphi(h)))}{\ln x} = \lim_{x\to 0} \frac{\ln(\limsup_{h\to 0}(\varphi(hx)/\varphi(h)))}{\ln x},$$
$$M(\varphi) = \sup_{x>1} \frac{\ln(\limsup_{h\to 0}(\varphi(hx)/\varphi(h)))}{\ln x} = \lim_{x\to \infty} \frac{\ln(\limsup_{h\to 0}(\varphi(hx)/\varphi(h)))}{\ln x} \tag{A.2}$$

are known as *the Matuszewska-Orlicz type lower and upper indices* of the function $\varphi(r)$. The property of functions to be almost increasing or almost decreasing after the multiplication (division) by a power function is closely connected with these indices. We refer to [22–28] for such a property and these indices.

Note that in this definition $\varphi(x)$ need not to be an N-function: only its behaviour at the origin is of importance. Observe that $0 \le m(\varphi) \le M(\varphi) \le \infty$ for $\varphi \in W_0$, and $-\infty < m(\varphi) \le M(\varphi) \le \infty$ for $\varphi \in \overline{W}$, and the following formulas are valid:

$$m[x^a\varphi(x)] = a + m(\varphi), \quad M[x^a\varphi(x)] = a + M(\varphi), \quad a \in \mathbb{R}^1, \tag{A.3}$$

$$m([\varphi(x)]^a) = am(\varphi), \quad M([\varphi(x)]^a) = aM(\varphi), \quad a \ge 0, \tag{A.4}$$

$$m\left(\frac{1}{\varphi}\right) = -M(\varphi), \qquad M\left(\frac{1}{\varphi}\right) = -m(\varphi), \tag{A.5}$$

$$m(uv) \ge m(u) + m(v), \qquad M(uv) \le M(u) + M(v) \tag{A.6}$$

for $\varphi, u, v \in \overline{W}$.

The proof of the following statement may be found in [21], Theorems 3.1, 3.2, and 3.5. (In the formulation of Theorems in [21] it was supposed that $\beta \ge 0$, $\gamma > 0$ and $\varphi \in W_0$. It is evidently true also for $\varphi \in \overline{W}$ and all $\beta, \gamma \in \mathbb{R}^1$, in view of formulas (A.3)).

Theorem A.3. *Let $\varphi \in \overline{W}$ and $\beta, \gamma \in \mathbb{R}^1$. Then $\varphi \in \mathbb{Z}^\beta \Leftrightarrow m(\varphi) > \beta$ and $\varphi \in \mathbb{Z}_\gamma \Leftrightarrow M(\varphi) < \gamma$. Besides this, $m(\varphi) = \sup\{\mu > 0 : \varphi(x)/x^\mu$ is almost increasing$\}$, and $M(\varphi) = \inf\{\nu > 0 : \varphi(x)/x^\nu$ is almost decreasing$\}$, and for $\varphi \in \Phi_\gamma^\beta$ the inequalities*

$$c_1 x^{M(\varphi)+\varepsilon} \le \varphi(x) \le c_2 x^{m(\varphi)-\varepsilon} \tag{A.7}$$

hold with an arbitrarily small $\varepsilon > 0$ and $c_1 = c_1(\varepsilon)$, $c_2 = c_2(\varepsilon)$.

We define the following subclass in $\overline{W}_0$:

$$\overline{W}_{0,b} = \left\{\varphi \in \overline{W}_0 : \frac{\varphi(t)}{t^b}, \text{ is almost increasing}\right\}, \quad b \in \mathbb{R}^1. \tag{A.8}$$

A.2. ZBS Classes and MO Indices of Weights at Infinity

Following Section 2.2 in [29], we use the following notation.

Let $-\infty < \alpha < \beta < \infty$. We put $\Psi_\alpha^\beta := \widehat{\mathbb{Z}}^\beta \cap \widehat{\mathbb{Z}}_\alpha$, where $\widehat{\mathbb{Z}}^\beta$ is the class of functions $\varphi \in \overline{W}_\infty$ satisfying the condition $\int_x^\infty (x/t)^\beta(\varphi(t)dt/t) \le c\varphi(x)$, $x \in (\ell,\infty)$, and $\widehat{\mathbb{Z}}_\alpha$ is the class of functions $\varphi \in W([\ell,\infty))$ satisfying the condition $\int_\ell^x (x/t)^\alpha(\varphi(t)dt/t) \le c\varphi(x)$, $x \in (\ell,\infty)$, where $c = c(\varphi) > 0$ does not depend on $x \in [\ell,\infty)$.

The indices $m_\infty(\varphi)$ and $M_\infty(\varphi)$ responsible for the behavior of functions $\varphi \in \Psi_\alpha^\beta([\ell,\infty))$ at infinity are introduced in the way similar to (A.2):

$$m_\infty(\varphi) = \sup_{x>1} \frac{\ln\left[\liminf_{h\to\infty}(\varphi(xh)/\varphi(h))\right]}{\ln x},$$
$$M_\infty(\varphi) = \inf_{x>1} \frac{\ln\left[\limsup_{h\to\infty}(\varphi(xh)/\varphi(h))\right]}{\ln x}. \tag{A.9}$$

Properties of functions in the class $\Psi_\alpha^\beta([\ell,\infty))$ are easily derived from those of functions in $\Phi_\beta^\alpha([0,\ell])$ because of the following equivalence

$$\varphi \in \Psi_\alpha^\beta([\ell,\infty)) \Longleftrightarrow \varphi_* \in \Phi_{-\alpha}^{-\beta}([0,\ell^*]), \tag{A.10}$$

where $\varphi_*(t) = \varphi(1/t)$ and $\ell_* = 1/\ell$. Direct calculation shows that

$$m_\infty(\varphi) = -M(\varphi_*), \qquad M_\infty(\varphi) = -m(\varphi_*), \qquad \varphi_*(t) := \varphi\left(\frac{1}{t}\right). \tag{A.11}$$

By (A.10) and (A.11), one can easily reformulate properties of functions of the class Φ_γ^β near the origin, given in Theorem A.3 for the case of the corresponding behavior at infinity of functions of the class Ψ_α^β and obtain that

$$c_1 t^{m_\infty(\varphi)-\varepsilon} \le \varphi(t) \le c_2 t^{M_\infty(\varphi)+\varepsilon}, \quad t \ge \ell,\ \varphi \in \overline{W}_\infty,$$
$$m_\infty(\varphi) = \sup\left\{\mu \in \mathbb{R}^1 :\ t^{-\mu}\varphi(t) \text{ is almost increasing on } [\ell,\infty)\right\}, \tag{A.12}$$
$$M_\infty(\varphi) = \inf\left\{\nu \in \mathbb{R}^1 :\ t^{-\nu}\varphi(t) \text{ is almost decreasing on } [\ell,\infty)\right\}.$$

We say that a continuous function φ in $(0,\infty)$ is in the class $\overline{W}_{0,\infty}(\mathbb{R}_+^1)$ if its restriction to $(0,1)$ belongs to $\overline{W}([0,1])$ and its restriction to $(1,\infty)$ belongs to $\overline{W}_\infty([1,\infty])$. For functions in $\overline{W}_{0,\infty}(\mathbb{R}_+^1)$ the notation

$$\mathbb{Z}^{\beta_0,\beta_\infty}\left(\mathbb{R}_+^1\right) = \mathbb{Z}^{\beta_0}([0,1]) \cap \mathbb{Z}^{\beta_\infty}([1,\infty)), \qquad \mathbb{Z}_{\gamma_0,\gamma_\infty}\left(\mathbb{R}_+^1\right) = \mathbb{Z}_{\gamma_0}([0,1]) \cap \mathbb{Z}_{\gamma_\infty}([1,\infty)) \tag{A.13}$$

has an obvious meaning (note that in (A.13) we use $\mathbb{Z}^{\beta_\infty}([1,\infty))$ and $\mathbb{Z}_{\gamma_\infty}([1,\infty))$, not $\widehat{\mathbb{Z}}^{\beta_\infty}([1,\infty))$ and $\widehat{\mathbb{Z}}_{\gamma_\infty}([1,\infty))$). In the case where the indices coincide, that is, $\beta_0 = \beta_\infty := \beta$, we will simply write $\mathbb{Z}^{\beta}(\mathbb{R}^1_+)$ and similarly for $\mathbb{Z}_\gamma(\mathbb{R}^1_+)$. We also denote

$$\Phi^{\beta}_{\gamma}\left(\mathbb{R}^1_+\right) := \mathbb{Z}^{\beta}\left(\mathbb{R}^1_+\right) \cap \mathbb{Z}_{\gamma}\left(\mathbb{R}^1_+\right). \tag{A.14}$$

Making use of Theorem A.3 for $\Phi^{\alpha}_{\beta}([0,1])$ and relations (A.11), one easily arrives at the following statement.

Lemma A.4. *Let $\varphi \in \overline{W}(\mathbb{R}^1_+)$. Then*

$$\varphi \in \mathbb{Z}^{\beta_0,\beta_\infty}\left(\mathbb{R}^1_+\right) \Longleftrightarrow m(\varphi) > \beta_0, \quad m_\infty(\varphi) > \beta_\infty, \tag{A.15}$$

$$\varphi \in \mathbb{Z}_{\gamma_0,\gamma_\infty}\left(\mathbb{R}^1_+\right) \Longleftrightarrow M(\varphi) < \gamma_0, \quad M_\infty(\varphi) < \gamma_\infty. \tag{A.16}$$

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