The all-pay auction with cross-shareholdings

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Abstract

We consider an all-pay auction between several firms under asymmetric information in which each firm owns a share in its rival. We characterize the equilibrium and show how much these cross-shareholdings serve to dampen competition. Additionally, we explain why the well known relationship between the equilibrium strategies of the standard first price and all-pay auctions breaks down in our setting.
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1 Introduction

Share crossholdings change firms' competitive incentives. This has been observed early on in the context of Cournot competition by Reynolds and Snapp (1986). The implications of crossholdings of shares have been analysed in the context of winner-pay auctions by Dasgupta and Tsui (2004) and Greenlee and Waehrer (2004). A further mode of competition occurs in beauty contests and lobbying games in which all competitors pay their bids. The analysis here is the first to study the role of share crossholdings for all-pay auctions in a framework with incomplete information. We focus on symmetric independent valuations, and compare bidding behaviour in the all-pay auction with the corresponding standard (winner-pay) first-price auction.

2 The all-pay auction

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There are $N = \{1, 2, ... n\}$ firms who each draw their valuation of winning the all-pay auction independently from a continuous, strictly increasing cumulative distribution function $F(v)$ with density $f(v)$. The support is $[0, \overline{v}]$. The actual draw is private information. Firms make bids x_i and the firm with the highest bid is awarded the prize. With multiple highest bids, the prize is randomly allocated among the firms making the highest bid. All firms must pay their bids in full, irrespective of who wins the prize.

Each firm's operating profit equals the probability of winning the auction times the firm's valuation v_i minus the firm's bid x_i. Firms maximize their operating profits in the absence of cross-holdings. Cross-holdings change firms' objective functions. As shown by Greenlee and Waehrer (2004), the reduced form of their objective functions becomes a weighted sum of all firms' operating profits, with weights being functions of the crossholdings of shares.

With n firms there are many cross-ownership constellations. In order to derive analytical solutions we adopt the notion of semi-symmetry of Greenlee and Waehrer. Let θ_{ii} be the weight that firm i attributes to its own operating profit, and θ_{ij} be the corresponding weight by which the operating profit of firm j enters into firm i's objective function. Define the average weight i attributes to the other firms' operating profits as $\overline{\theta}_i = \frac{1}{n-1} \sum_{j \in N \setminus \{i\}} \theta_{ij}$. Ownership shares are semi-symmetric if for all $i, j \in N$, $\theta_{ii} = \theta_{jj}$ and $\overline{\theta}_i = \overline{\theta}_j$. We assume that a firm cares more about its own operating profits than about those of rivals so that $\theta_{ii} > \overline{\theta}_i$ for all i.

Let $g(v) = x$ be a monotonically increasing, differentiable bidding strategy; its inverse is given by $g^{-1}(x) = v$. Denote by $F_{n-1}(t)$ the distribution of the highest valuation among $i's$ opponents, and by $f_{n-1}(t)$ its density. Suppose further that all players other than i have adopted this strategy, and consider the objective function of firm i with valuation v and bid/effort x.

$$
\begin{aligned}
\pi_i(v, x) &= \theta_{ii} \Big(\int_0^{g^{-1}(x)} v f_{n-1}(t) dt - x \Big) \\
&\quad + \overline{\theta}_i \int_{g^{-1}(x)}^{\overline{v}} t f_{n-1}(t) dt - \overline{\theta}_i (n-1) \int_0^{\overline{v}} g(t) f(t) dt
\end{aligned}
\tag{1}
$$

The first component in (1) is the weighted expected operating profit. It wins if it beats all $(n-1)$ opponents with value-bid pair (v, x), and the second is the weighted average share in the other firms' expected prize value if one of these should win (given that i does not know the other's exact valuation). The third term represents $i's$ share of the bid costs of the competitors through cross-ownership.

Differentiating (1) with respect to the effort x gives (here $'$ denotes derivative)

$$
\frac{\partial \pi_i(v, x)}{\partial x} = \theta_{ii}[v f_{n-1}(g^{-1}(x)) g^{-1'}(x) - 1] - \overline{\theta}_i [g^{-1}(x)(f_{n-1}(g^{-1}(x)) g^{-1'}(x)]
\tag{2}
$$

Writing $g^{-1}(x) = v$ by definition and noting that $g^{-1'}(x) = \frac{1}{g'(g^{-1}(x))}$ means that (2) can be written as

$$\frac{\partial \pi_i(v, x)}{\partial x} = \theta_{ii} \left[\frac{v f_{n-1}(v)}{g'(v)} - 1 \right] - \overline{\theta}_i \left[\frac{v f_{n-1}(v)}{g'(v)} \right] \tag{3}$$

which must equal zero at an optimum. Setting $\frac{\partial \pi_i(v, x)}{\partial x} = 0$ and rearranging gives

$$g'(v) = v f_{n-1}(v)(\alpha v + (1 - \alpha)g(v)) \tag{4}$$

where $\alpha \equiv 1 - \frac{\overline{\theta}_i}{\theta_{ii}}$. Note that the equilibrium is homogenous of degree zero in $\overline{\theta}_i$ and θ_{ii}; hence the symmetry of the model is a restriction on the composition of ownership (in own firm versus cross ownership), not a restriction on individual ownership shares. Solving this differential equation yields:

$$g(v) = \alpha \int_0^v t v f_{n-1}(t) dt + K \tag{5}$$

where K is a constant of integration which equals zero since $g(0) = 0$. Integrating the right-hand-side of (5) by parts gives the bidding strategy in the symmetric equilibrium of the all-pay auction with cross-shareholdings as

$$g(v) = \alpha \left[v F_{n-1}(v) - \int_0^v F_{n-1}(t) dt \right] \tag{6}$$

When $\overline{\theta}_i = 0$, then $\alpha = 1$ and the bid strategy is of course the same as in the regular all-pay auction under asymmetric information. Notice that as $\overline{\theta}_i$ increases then the bid of each player is reduced to a fraction of its level in the absence of cross-shareholdings. Hence the expected revenue from the all-pay auction is a fraction $\alpha(\overline{\theta}_i)$ of its level without cross-shareholdings.

3   The standard first-price auction

In the all-pay auction, the cross-shareholding simply reduces the size of the bid by a factor $\alpha(\overline{\theta}_i)$, but in the standard (winner-pay) first-price auction this has a more fundamental effect on the equilibrium bid strategies. Writing $b(v)$ as the symmetric bid function and taking equilibrium behaviour of other firms as given, the objective function of firm i in the standard first-price auction can be written

$$\pi_i^{fp}(v, x) = \theta_{ii} \int_0^{b^{-1}(x)} (v - x) f_{n-1}(t) dt + \overline{\theta}_i \int_{b^{-1}(x)}^{\overline{v}} (t - b(t)) f_{n-1}(t) dt \tag{7}$$

The first order condition

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$$\frac{\partial \pi_i^{fp}(v,x)}{\partial x} = (\theta_{ii} - \overline{\theta}_i)(v - b(v))\frac{f_{n-1}(v)}{b'(v)} - \theta_{ii}F_{n-1}(v) = 0 \qquad (8)$$

can be rearranged to yield the first order differential equation

$$b'(v) = \alpha(v - b(v))\frac{f_{n-1}(v)}{F_{n-1}(v)}$$

Thus the equilibrium strategy $b(v)$ in the n-player standard first-price auction with cross-shareholding (and a reserve price of zero) is

$$b(v) = v - \int_0^v \left(\frac{F_{n-1}(t)}{F_{n-1}(v)}\right)^\alpha dt \qquad (9)$$

a result first demonstrated by Greenlee and Waehrer (2004). One can see that when $\overline{\theta}_i = 0$ (i.e. $\alpha = 1$) we have the usual relationship that $g(v) = F(v)_{n-1}b(v)$, i.e. the bid in the all-pay auction is the same as the expected payment by type v in the standard first-price auction. When $\overline{\theta}_i$ approaches θ_{ii}, that is α approaches zero, equilibrium bids converge to zero in the standard first-price auction as in the all-pay auction. The case α equal to zero implies that the interests of all players are aligned in the sense that they have a common interest in allocating the object to the bidder with the highest valuation. With interests being fully aligned, the game reduces to a pure coordination game, with a separating equilibrium in a small interval at zero. Due to separation the object is allocated to the firm with the highest valuation.

Furthermore, $b(v)$ is strictly concave in α. Since $F(v)_{n-1}b(v)$ equals $g(v)$ when α equals zero and one, and $g(v)$ increases linearly in α, it follows that $F(v)_{n-1}b(v)$ strictly exceeds $g(v)$ for all α except at the boundaries. Hence, the ex ante expected revenue per player from the standard first-price auction $ER^{FPA} = \int_0^{\overline{v}} F(t)_{n-1}b(t)f(t)dt$ strictly exceeds the expected revenue per player in the all-pay auction $ER^{APA} = \int_0^{\overline{v}} g(t)f(t)dt$.

To provide intuition on the more aggressive bidding behavior in the first price auction, consider the effect of a change in cross ownership on the marginal benefit of bidding. Differentiating the two first order conditions (3) and (8) with respect to $\overline{\theta}_i$, yields

$$\frac{\partial^2 \pi_i}{\partial x \partial \overline{\theta}_i} = -v\frac{f_{n-1}(v)}{g'(v)} \text{ and}$$

$$\frac{\partial^2 \pi_i^{fp}}{\partial x \partial \overline{\theta}_i} = -[v - b(v)]\frac{f_{n-1}(v)}{b'(v)}$$

Here $f_{n-1}(v)/g'(v)$ is the increase in win probability associated with a higher bid. In the all-pay auction, where firms pay their bids unconditionally, the total effect on revenue flows from ownership shares in rival firms is proportional to *gross* value v. In the standard first-price auction, however, competing firms

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carry no bidding costs in case the firm wins the auction. Thus, the effect on revenue flows is proportional to *net* value $v - b(v)$. In other words, by increasing the bid in the standard first-price auction, a firm reduces its expected external bidding cost, which is a relevant saving. This effect has no correspondence in the all-pay auction, and explains the more aggressive bidding that occurs in the standard first-price auction.

4   References

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