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Insulating the Vacuum

Calculating the Casimir force using the boundary integral method with von Neumann boundary conditions

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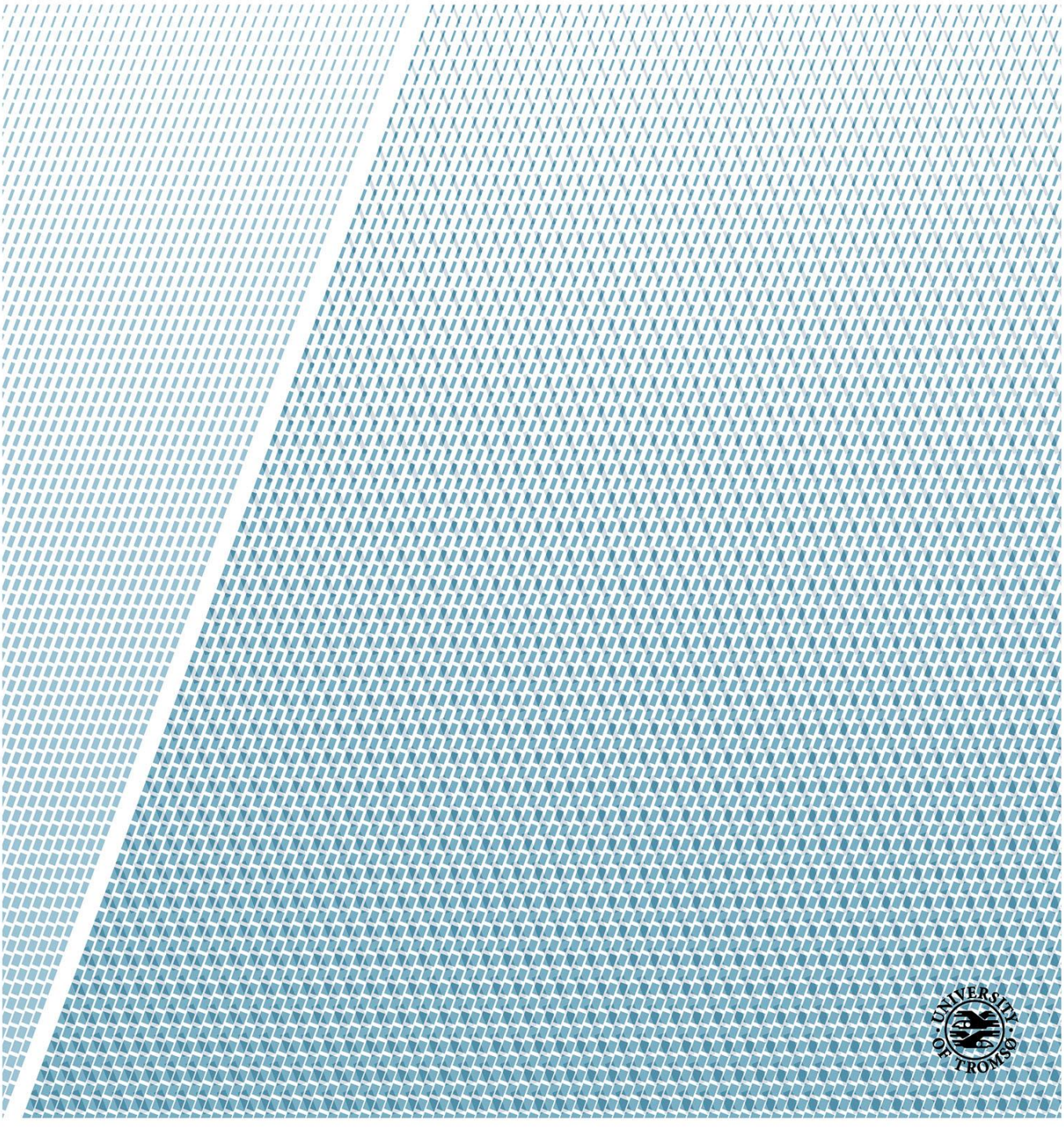

Abstract

In 2012, a new method for calculating the Casimir force between compact objects was developed [14, 15], expressing the force in terms of a boundary integral equation. The case of perfectly conducting objects with Dirichlet boundary conditions in two dimensions was treated in [14]. The method was later extended to three dimensions [20].

The contribution of this thesis will be to develop the method in two dimensions for the case when the objects are perfectly insulating, meaning von Neumann boundary conditions. A formula for the Casimir force in terms of a boundary integral problem is derived and shown to correctly predict the force between two parallel plates, except for a missing factor of 2 that was also observed for Dirichlet boundary conditions. The developed formula contains a coefficient that is dependent on the regularization scheme used, and it is not clear whether this coefficient is geometry-independent.

Dedicated to Jiajia

Acknowledgements

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Contents

Chapter 1

Introduction

1.1 The Casimir effect

In 1948, Hendrik Casimir predicted that there would be an attractive force between uncharged, parallel conducting metal plates in vacuum at zero temperature [4]. His prediction was that the force per area at a separation distance a was

$$\frac{F(a)}{A} = -\frac{\pi^2}{240}\frac{\hbar c}{a^4}. \tag{1.1}$$

In a classical sense, this is a surprising result; the plates are uncharged and placed in a vacuum, so there is nothing that could produce such forces.

While the Casimir effect usually refers to the quantum physical phenomenon, analogous phenomena occur in different kinds of wave physics. A similar phenomenon is observed when transverse waves move through a loaded spring [7], i.e. N beads linked together and allowed to oscillate in one dimension. It has been shown that placing two beads on a string driven by transverse oscillations will experience a force akin to the Casimir force, which can be either attractive or repulsive depending on the frequency of the waves [13]. Another example is that plates suspended in a medium such as a water are forced together by acoustic waves in the fluid [9], and it has even been suggested that ships on a stormy sea will experience Casimir-like effects [2].

A simplified explanation is that the area between the objects is sheltered and there is low wave activity there, leading to an external pressure from the areas where wave activity is high. It seems reasonable that part of the explanation behind the Casimir effect is a more general wave phenomenon.

The mystery of the Casimir force is then the question, where do the waves come from? We are talking about uncharged plates in vacuum, after all. The typical explanation is that in quantum field theory and quantum electrodynamics in particular, what we classically think of as "vacuum" is actually the lowest possible energy state, the ground state, of the electromagnetic field. There is no such thing as an absolute void, similar to how there is still water in a lake even when there are no waves. This field has ripples known as vacuum fluctuations, that occur randomly in accordance with Heisenberg's uncertainty principle. When the plates are introduced into this field, they cause a disturbance in the field, and this interaction is what gives rise to the Casimir effect.

1.2 History

Casimir and Polder predicted this effect in 1948 [5], describing the interaction between a perfectly conducting plate and a particle, and between two particles. Their original interpretation of this force was as a retardation effect on van der Waals forces. Later the same year, Casimir published a paper discussing the case of two perfectly conducting plates [4]. The first measurement was attempted by Sparnaay et al. in 1958, but the errors gave a 100% uncertainty. They could only conclude that "the observed attractions do not contradict Casimir's theoretical prediction" [26].

After this, the 1948 paper went unnoticed for a long time, and started getting attention in the 1970s. Schwinger described the effect in 1975 in terms of source theory, thereby explaining it without reference to vacuum fluctuations [25]. This has interesting metaphysical implications, as it shows that the vacuum energy of the zero-point fluctuations is not a prerequisite for the Casimir effect. Conversely, it shows that a measurement of the Casimir effect is not necessarily evidence of the reality of zero-point fluctuations.

In 1997, S.K. Lamoreaux measured the force between a plate and a sphere, and found the results to be within 5% of the predicted values [17]. This is regarded as the first successful measurement, almost 50 years after the first prediction. Mohideen and Roy also measured the force between a plate and a sphere in 1998, and the results differed by less than 2% from the theoretical prediction [21].

1.3 Applications

Equation (1.1) shows that the force between two plates drops off quickly as the separation distance increases. On the other hand, making reference to plates and spheres suggests a macroscopic structure that is not clearly seen on an atomic level, and moreover, the length scales must be larger than the penetration depth of the material in the plates. We find that Casimir forces are significant on length scales of about 0.1-1 μm; on smaller scales, the van der Waals forces dominate, and on larger scales, the forces become insignificant.

With nanotechnology on the rise and the development of microscopic devices such as sensors, routers, atuators, accelerometers and microphones, the Casimir force is becoming more and more significant in engineering. One problem of microelectromechanical (MEMS) devices is a phenomenon called stiction. Moving components of MEMS devices frequently move into contact with fixed electrodes and stick to them, leading to loss of functionality. It has been recognized that the Casimir force is a primary cause for this phenomenon [3]. Better understanding of the Casimir force can help us avoid this problem.

The repulsive Casimir forces can potentially be taken advantage of, for example through quantum levitation of objects leading to new devices with ultra-low static friction [22]. The fact that the Casimir effect can be both beneficial and harmful demonstrates the importance of understanding it, and motivates further theoretical and experimental study.

1.4 Calculating the Casimir force

The original way Casimir calculated the energy in his 1948 paper was through mode summation. The method decomposes the electromagnetic field into an infinite number of harmonic oscillators called modes, where the nth mode has frequency ω_n. The total energy is found by summing the energies from each mode,

$$E = \sum_n \frac{1}{2}\hbar\omega_n. \tag{1.2}$$

This method has evolved since Casimir first applied it, and modern approaches use methods like the argument principle and zeta function regularization. This method gives an exact result, but it only works for very special configurations.

The problem is that one has to find the whole frequency spectrum, $\{\omega_n\}$. This is difficult at best, and normally impossible to do analytically for all but the most symmetric cases. Furthermore, the sum generally diverges and regularization must be used to extract a finite expression for the energy, and the the mathematical framework for this process is not comprehensive. Typical examples where mode summation can be used are parallel plates, coaxial cylinders, and concentric spheres. In the case of concentric spheres, the applicability of the method is already stretched to its limits. Despite these drawbacks, the method still has theoretical advantages. In particular, when developing new numerical methods, it is vital to test the methods by applying them to configurations where the mode summation has given an exact answer.

An early method for calculations on non-planar configurations is the proximity force approximation (PFA). For a long time, this was the only practical way to calculate the Casimir effect for configurations other than parallel plates [19]. The idea of this method is to approximate curved surfaces as flat, and treat interacting objects as a set of pairs of small parallel plates. The plates are extended to infinity and Casimir's result is applied to each of the pairs, then the Casimir energy is found by summing the contributions from each of the pairs. The major limitation of PFA is that treating interacting points as a pair of parallel plates is applicable only to very small separations.

Over the last decade, there has been developed methods that don't require information about the mode spectrum. One such method is what we refer to as the functional integral method. The method was first developed by T. Emig et al. [11] and was further developed by I. Kilen and P. Jakobsen [14], and is based on Feynman's idea of integrating over weighted classical paths. The method relates the Casimir energy to a functional integral of Gaussian, which expresses the energy in terms of the determinant of a finite matrix. This method has been very successful for calculating the Casimir energy in a wide variety of situations. However, since it requires calculation of a determinant of a matrix that can become very large, it is hard to make an efficient implementation, and it is not easily parallelized.

In 2006, another numerical approach was introduced, using Green's functions based on the finite difference time domain method (FDTD) from computational electromagnetics [24]. This method calculates the Casimir force directly for complex geometries. Since all the aforementioned methods output the Casimir energy and since energy is related to force via a gradient, two evaluations of the energy are required to calculate the force vector. However,

when applying FDTD, the whole space is discretized and the Green's function is calculated at each point in space, including space between the objects. In the end, only the value of the Green's functions on the surfaces of the objects is of importance, and this seems to indicate that the method requires unnecessarily many calculations.

Another method using Green's functions (and the one we will primarily work with in this thesis) is the Boundary Integral Method (BIM), introduced by I. Kilen and P. Jakobsen in 2012 [14] [15], and extended by K. Mikalsen in 2014 [20]. This method applies to arbitrary configurations and is most efficient when applied on linear equations and piecewise linear material coefficients. The Casimir pressure is expressed in terms of a boundary integral problem[†]. The BIM has an advantage over FDTD in that it only calculates the pressure on boundaries. Computationally, it is based on filling and solving a set of linear equations, which is more easily parallelized than FIM. Another advantage of the BIM is that the integral equations can be regularized in a geometry-independent fashion, thereby making the results more easily generalized.

1.5 Notation and governing equations

We begin our discussion with the Lagrangian for a massless scalar field φ in vacuum,

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi. \tag{1.3}$$

This is the basic principle from which all of our discussion springs; its derivation is beyond the scope of this thesis. Note that this is a quantum mechanical phenomenon, as in a classical vacuum there would be no field, i.e. $\mathcal{L}_{\text{classical}} = 0$. The Euler-Lagrange equations,

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) = 0, \tag{1.4}$$

give us the wave equation

$$\varphi_{tt} - \nabla^2\varphi = 0, \tag{1.5}$$

which forms the basis for several of the techniques discussed. We will use natural units throughout, setting $\hbar = c = 1$.

[†]A boundary integral problem normally takes the form $\kappa f(\boldsymbol{x}) = v(\boldsymbol{x}) + \int_S dA\, K(\boldsymbol{x})f(x)$, where α is constant and V and K are known functions, and asks you to solve for $f(\boldsymbol{x})$.

Throughout the thesis we will consider a situation of r disjoint objects in d-dimensional space. We always denote the volumes filled by these objects by $V_1, \ldots, V_r$, and let $Q_j = \partial V_j$ be their boundaries. We use V_0 to denote the exterior of all plates, $Q = \bigcup_{j=1}^{r} Q_j$ to denote the total boundary of all the objects with normals pointing into V_0. Integration over Q can in general be written as

$$\int_Q dA = \sum_\gamma \int_{Q_\gamma} dA_\gamma. \tag{1.6}$$

Sometimes we also use ∂V_0 to indicate the boundary of V_0, which is equal to Q except the normals point into the objects (when unspecified, the surface normal implicitly points out of the objects).

We will often speak about block matrices. If we say that M is a block matrix with entries $M_{\alpha\beta}^{ij}$, we mean that

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1r} \\ M_{21} & M_{22} & \cdots & M_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ M_{r1} & M_{r2} & \cdots & M_{rr} \end{bmatrix}, \quad M_{\alpha\beta} = \begin{bmatrix} M_{\alpha\beta}^{11} & M_{\alpha\beta}^{12} & \cdots & M_{\alpha\beta}^{1N} \\ M_{\alpha\beta}^{21} & M_{\alpha\beta}^{22} & \cdots & M_{\alpha\beta}^{2N} \\ \vdots & \vdots & \ddots & \vdots \\ M_{\alpha\beta}^{N1} & M_{\alpha\beta}^{N2} & \cdots & M_{\alpha\beta}^{NN} \end{bmatrix}. \tag{1.7}$$

We say that $M_{\alpha\beta}$ are the matrix entries of M. If the block matrix consists of a single column where each entry is a column vector, we sometimes refer to it as a block vector.

For perfectly conductive plates, there should be no field on the boundaries, i.e.

$$\varphi|_Q = 0. \tag{1.8}$$

These are called Dirichlet boundary conditions. Kilen first developed BIM for this case in $d = 2$ dimensions, and Mikalsen extended it to $d = 3$ dimensions. In this thesis, we will consider the case when the plates are perfectly insulating, which is mathematically expressed as

$$\partial_n \varphi|_Q = 0, \tag{1.9}$$

where ∂_n is the normal derivative at the point on Q. We will develop BIM for $d = 2$ dimensions.

An obstacle that always seems to come up when calculating the Casimir effect is infinite expressions (for example, for the BIM, we find that the Green's functions are singular on the boundaries, and often extremely so). The process of removing these infinities is known as regularization.

When calculating Casimir energy, one common regularization step is based on the idea that the free vacuum with no objects already contains infinite, constant energy E_∞. Bringing in objects creates finite fluctuations $\tilde{E}(\boldsymbol{x}, t)$, and since the dynamics of the system are described in terms of the gradient of energy, these fluctuations fully describe the observable physics. Mathematically, if the total energy is given by $E(\boldsymbol{x}, t) = E_\infty + \tilde{E}(\boldsymbol{x}, t)$, then $\nabla E = \nabla \tilde{E}$. Therefore, a regularization step often consists of calculating the energy when the boundaries vanish, then subtracting this value from the original energy.

When working with Casimir forces, a common regularization step is to recognize that the directly obtained pressure on an object is the sum of two parts: the interaction pressure arising from the proximity of two objects, and the self pressure interpreted as an objects pressure on itself. The self pressure cannot contribute to the net force on an object, as this would violate conservation of momentum. If the self pressure is infinite and the interaction pressure is finite, the problem will be about how to extract the interaction pressure.

1.6 Layout and aims

The main objective of this thesis is to develop the BIM with von Neumann boundary conditions in two dimensions.

Preliminarily, in Chapter 2 we will look at ways of obtaining exact expressions for the energy. We consider two cases: parallel plates and concentric circles. As we study these cases, especially the case of concentric circles, the difficulty of these calculations shall be made apparent. It helps us appreciate the notion that it is virtually impossible to acquire exact expressions for more complicated situations, and the necessity for numerical methods.

In Chapter 3 we study the existing functional integral method. Although a powerful method, the mathematical framework is very complicated.

In Chapter 4, we develop the boundary integral method for von Neumann conditions. This chapter contains most of the original work of this thesis. We consider examples of the parallel plates and concentric circles.

Chapter 2

Special cases with exact solutions

In this section we will calculate exact expressions for the Casimir force for the two special cases of parallel plates and concentric circles. Developing these exact expressions will be important later because it provides a baseline for testing our methods.

The problem we want to solve is

$$\nabla^2\varphi(\boldsymbol{x},t) - \varphi_{tt}(\boldsymbol{x},t) = 0 \tag{2.1a}$$

$$\partial_n\varphi|_Q = 0 \tag{2.1b}$$

where Q is the boundaries of all the objects. Taking the Fourier transform in time gives

$$\nabla^2\varphi(\boldsymbol{x},t) + \omega^2\varphi(\boldsymbol{x},t) = 0. \tag{2.2}$$

The boundary conditions will give a set of admissible frequencies, $\{\omega_n\}$, and the energy of the system is given in terms of the positive resonance frequencies as

$$E = \frac{1}{2}\sum_n \omega_n. \tag{2.3}$$

This sum generally diverges, so much of our work will be about regularizing it.

First we will consider the first special case of two infinitely long parallel plates at a separation distance a. In d dimensions, we consider the volumes to be

$$\begin{aligned} V_0 &= \left\{(x_1,\ldots,x_d) \in \mathbb{R}^d \,\middle|\, 0 < x_1 < a\right\} \\ V_1 &= \left\{(x_1,\ldots,x_d) \in \mathbb{R}^d \,\middle|\, x_1 < 0\right\} \\ V_2 &= \left\{(x_1,\ldots,x_d) \in \mathbb{R}^d \,\middle|\, x_1 > a\right\} \end{aligned} \quad , \tag{2.4}$$

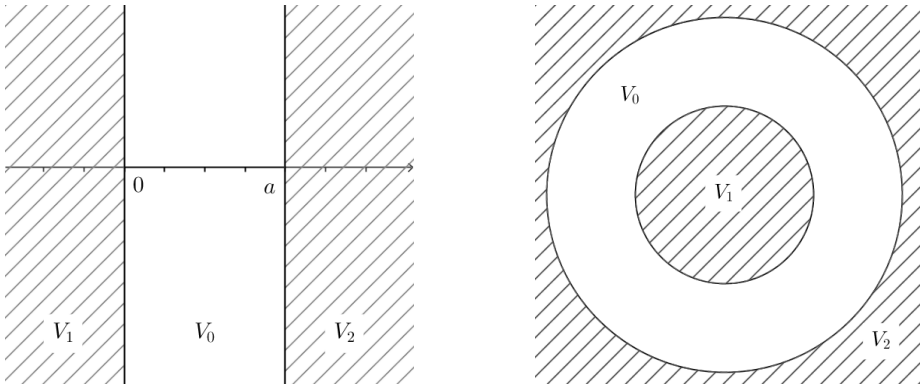
with normals pointing into the vacuum between the plates, as shown in Figure 2.1a. In Section 2.1, Section 2.2 and Section 2.3, we consider three ways of regularizing this case.

The second configuration we will consider are concentric spheres. The objects are concentric spheres of radii r_1 and r_2. The volumes are

$$\begin{aligned} V_0 &= \left\{ \boldsymbol{x} \in \mathbb{R}^d \,\middle|\, r_1 < \|x\| < r_2 \right\} \\ V_1 &= \left\{ \boldsymbol{x} \in \mathbb{R}^d \,\middle|\, \|\boldsymbol{x}\| < r_1 \right\} \\ V_2 &= \left\{ \boldsymbol{x} \in \mathbb{R}^d \,\middle|\, \|\boldsymbol{x}\| > r_2 \right\} \end{aligned} \tag{2.5}$$

as shown in Figure 2.1b. This case is treated in Section 2.4.

Finally in Section 2.5 we will describe a way to calculate pressure from the energy. In addition to the fact that knowing pressure is useful on its own, it will be particularly important for us later as BIM gives pressure directly and we must establish how to compare this to the exact answer.



(a) Parallel plates **(b)** Concentric circles

Figure 2.1: The configurations of parallel plates and concentric circles. The shaded area represents the interiors of the objects, and the white areas represent vacuum.

2.1 Parallel plates using vacuum energy regularization

In this section we study the parallel plates in a fashion similar to the way first described by Casimir [4]. He treated three dimensional space with

perfectly conducting boundaries. Here, on the other hand, we will consider one-dimensional space with two perfectly insulating plates at a distance a. A one-dimensional space significantly simplifies the mathematical machinery, but retains same physical arguments as those Casimir originally gave, enabling us to highlight the parts that are important for our discussion.

The problem (2.1) in one dimension can be written as

$$\varphi_{xx} - \varphi_{tt} = 0 \tag{2.6}$$

$$\varphi_x(0) = \varphi_x(a) = 0. \tag{2.7}$$

By Fourier transforming time, we get

$$\varphi''(x) + \omega^2 \varphi = 0, \tag{2.8}$$

which has general solutions

$$\varphi(x) = A\cos\omega x + B\sin\omega x. \tag{2.9}$$

The first boundary condition gives $B = 0$, and then the second boundary condition becomes

$$-\omega A \sin\omega a = 0. \tag{2.10}$$

We must have $A \neq 0$ or φ vanishes, so the valid resonance frequencies are

$$\omega_n = \frac{n\pi}{a}. \tag{2.11}$$

Then the energy is given by

$$E(a) = \frac{1}{2}\sum_{n=1}^{\infty}\omega_n = \frac{1}{2}\sum_{n=1}^{\infty}\frac{n\pi}{a}. \tag{2.12}$$

This sum is divergent and must be regularized. To do this, we subtract the energy when the plates are moved to infinite separation E_∞. Let $\Delta\omega = \pi/a$, so that in the limit $a \to \infty$, we get

$$E_\infty = \frac{1}{2}\frac{1}{\Delta\omega}\sum_{n=1}^{\infty}\Delta\omega\,\omega_n = \frac{a}{2\pi}\int_0^\infty d\omega\,\omega. \tag{2.13}$$

This integral is also divergent, and another part of our regularization scheme is to determine how to calculate the difference $\mathcal{E} = E - E_\infty$.

The idea is to cut off the high frequency contributions to both sides to get a finite answer, then take the difference, and finally let the cutoff go to infinity. As Casimir himself put it, "for very short waves (X-rays e.g.) our plate is hardly an obstacle at all and therefore the zero point energy of these waves will not be influenced by the position of this plate." [4]. To do this, multiply the summands in (2.12) by a function $f_\varepsilon(\omega_n)$ which is unity for small ω_n, tends to zero as $\omega_n \to \infty$, and that converges to 1 pointwise when $\varepsilon \to 0$. We shall choose $e^{-\varepsilon\omega_n}$. Then (2.12) becomes

$$\tilde{E}(a) = \frac{1}{2}\sum_{n=1}^{N} \frac{\pi}{a} n e^{-\omega_n \varepsilon}. \tag{2.14}$$

Using the result

$$\sum_{n=1}^{\infty} n e^{-cn} = \frac{e^c}{(1-e^c)^2} \tag{2.15}$$

and the appropriate Taylor series, it can be shown that

$$\tilde{E}(a) = \frac{\pi}{2a}\frac{e^{\varepsilon\pi/a}}{(1-e^{\varepsilon\pi/a})^2} = \frac{a}{2\pi}\frac{1}{\varepsilon^2} - \frac{\pi}{24a} + \mathcal{O}(\varepsilon^2). \tag{2.16}$$

In the limit $a \to \infty$, the integral becomes

$$\tilde{E}_\infty = \frac{a}{2\pi}\int_0^\infty d\omega\, \omega e^{-\omega\varepsilon} = \frac{a}{2\pi}\frac{1}{\varepsilon^2}, \tag{2.17}$$

and therefore the regularized energy is

$$\mathcal{E}(a) = \lim_{\varepsilon\to 0}\left(\tilde{E}(a) - \tilde{E}_\infty\right) = -\frac{\pi}{24a}. \tag{2.18}$$

2.2 Parallel plates using zeta function regularization

The problem (2.1) can be written as

$$\begin{aligned} \varphi_{xx} + \nabla_{\boldsymbol{z}}^2\varphi - \varphi_{tt} &= 0 \\ \varphi_x(0,\boldsymbol{z},t) = \varphi_x(L,\boldsymbol{z},t) &= 0 \end{aligned} \tag{2.19}$$

where we view φ as a function of the special coordinate x perpendicular to the plates and the parallel components $\boldsymbol{z} = (x_2, \ldots, x_N)$. That is,

$$\varphi = \varphi(x, \boldsymbol{z}, t) \tag{2.20}$$

and the $\boldsymbol{z}$ Laplacian is

$$\nabla_{\boldsymbol{z}}^2 = x_2^2 + \cdots + x_N^2. \tag{2.21}$$

In the one-dimensional case there are no parallel components. We simply have $\varphi = \varphi(x, t)$ and $\nabla_z^2 = 0$.

By Fourier transforming time and the parallel coordinates, the time derivatives transform as $\partial_{tt} \to -\omega^2$ and the parallel Laplacian becomes $\nabla_{\boldsymbol{z}}^2 \to -k^2$, where

$$\boldsymbol{k} = (k_2, \ldots, k_N) \tag{2.22}$$

is a vector of the spatial frequencies of each coordinate. Then we get

$$\begin{aligned} \varphi_{xx} + \lambda^2 \varphi &= 0 \\ \varphi'(0) = \varphi'(a) &= 0 \end{aligned} \tag{2.23}$$

where $\lambda^2 = \omega^2 - k^2$. The general solutions are

$$\varphi(x) = A \sin \lambda x + B \cos \lambda x. \tag{2.24}$$

The boundary condition $\varphi'(0) = 0$ implies that $A = 0$. Then the second condition gives

$$\varphi'(a) = \lambda B \sin \lambda a = 0. \tag{2.25}$$

If $B = 0$, then $\varphi(x) = 0$ for all x, which is inadmissible. Therefore, we must have $\sin \lambda a = 0$, giving the eigenvalues

$$\lambda_n = \frac{n\pi}{a}. \tag{2.26}$$

The time-frequencies are

$$\omega_n(\boldsymbol{k}) = \sqrt{\left(\frac{n\pi}{a}\right)^2 + k^2} \tag{2.27}$$

and so the energy is

$$E = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \cdots \int_{-\infty}^{\infty} \frac{dk_d}{2\pi} \sum_{n=1}^{\infty} \omega_n(\boldsymbol{k}) = \frac{1}{2} \int_{\mathbb{R}^{d-1}} \frac{d\boldsymbol{k}}{(2\pi)^{d-1}} \sum_{n=1}^{\infty} \omega_n(\boldsymbol{k}). \tag{2.28}$$

This expression is divergent, so we have to regularize it. We will use the Riemann zeta function and analytic continuation to make sense of it.

The Riemann zeta function $\zeta(z)$ of a complex argument z is defined as

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z} \tag{2.29}$$

when $\Re(z) > 1$, and by analytic continuation elsewhere. By this notion, one can turn the expression around and associate the sum in (2.29) as the Riemann zeta function, even when the sum does not converge in the traditional sense. For example, with $z = -1$, we get

$$\sum_{n=1}^{\infty} n = -\frac{1}{12}. \tag{2.30}$$

This is of course perfectly absurd in the usual sense of a sum; we should understand the left-hand side as a "regularized sum", and try not to think of it as a limit of the partial sums.[†]

Write the energy as

$$E = \frac{\pi}{2a} \int_{\mathbb{R}^{d-1}} \frac{d\boldsymbol{k}}{(2\pi)^{d-1}} \sum_{n=1}^{\infty} \sqrt{n^2 + \left(\frac{ak}{\pi}\right)^2}. \tag{2.31}$$

Making the substitution $p_j = ak_j/\pi$, so that $\frac{d\boldsymbol{k}}{(2\pi)^{d-1}} = \frac{d\boldsymbol{p}}{(2a)^{d-1}}$, gives

$$E = \frac{\pi}{(2a)^d} \int_{\mathbb{R}^{d-1}} d\boldsymbol{p} \sum_{n=1}^{\infty} \sqrt{n^2 + p^2}. \tag{2.32}$$

The regularization consists of multiplying the summand by $(n^2 + p^2)^{-s}$ for some s that is large enough that the integral converges, then evaluating the resulting expression at $s = 0$. This process gives us the energy

$$\begin{aligned} E_s &= \frac{\pi}{(2a)^d} \int_{\mathbb{R}^{d-1}} d\boldsymbol{p} \sum_{n=1}^{\infty} \left(n^2 + p^2\right)^{\frac{1}{2}-s} \\ &= \frac{\pi}{(2a)^d} \int_{\mathbb{R}^{d-1}} d\boldsymbol{p} \sum_{n=1}^{\infty} n^{1-2s} \left(1 - \left(\frac{p}{n}\right)^2\right)^{\frac{1}{2}-s}. \end{aligned} \tag{2.33}$$

[†]This result is rather infamous, and is especially associated with S. Ramanujan who presented it in his letter to G. H. Hardy in 1913 [23].

Change the order of the sum and integration, and let $q_j = p_j/n$ to get

$$E_s = \frac{\pi}{(2a)^d} \sum_{n=1}^{\infty} n^{d-2s} \int_{\mathbb{R}^{d-1}} d\boldsymbol{q}\, (1+q^2)^{\frac{1}{2}-s}. \tag{2.34}$$

This separates the sum from the integral, and we can solve each separately. If $2s > d+1$, then the sum evaluates to

$$\sum_{n=1}^{\infty} n^{-(2s-d)} = \zeta(2s-d). \tag{2.35}$$

Thus, our regularized energy in d dimensions is

$$\mathcal{E} = \lim_{s\to 0} E_s = \frac{\pi}{(2a)^d} \lim_{s\to 0} \zeta(2s-d) \int_{\mathbb{R}^{d-1}} d\boldsymbol{q}\, \left(1+q^2\right)^{\frac{1}{2}-s}. \tag{2.36}$$

We will now evaluate this in the particular cases with 1, 2 and 3 dimensions.

2.2.1 One dimension

In 1 dimension, there is no integral and the energy is simply

$$\mathcal{E} = \frac{\pi}{2a}\zeta(-1) = -\frac{\pi}{24a}. \tag{2.37}$$

2.2.2 Two dimensions

In two dimensions, we must evaluate the integral

$$\int_{-\infty}^{\infty} dy\, (1+y^2)^{\frac{1}{2}-s}. \tag{2.38}$$

This integral converges for sufficiently large s, where its value is

$$\sqrt{\frac{\pi}{4}} \frac{\Gamma(s-1)}{\Gamma(s-\frac{1}{2})}. \tag{2.39}$$

Now, $\Gamma(-\frac{1}{2}) = -\sqrt{4\pi}$, but $\lim_{s\to 0} \Gamma(s-1)$ is infinite. However, $\zeta(-2) = 0$, so the energy might still be finite in the limit. Inserting this in (2.36),

$$\mathcal{E} = -\frac{\pi}{8a^2} \lim_{s\to 0} \zeta(2s-2)\Gamma(s-1). \tag{2.40}$$

The limit is

$$\lim_{s\to 0}\zeta(2s-2)\Gamma(s-1)=\frac{\zeta(3)}{2\pi^2} \tag{2.41}$$

so the energy is

$$\mathcal{E}=-\frac{\zeta(3)}{16\pi a^2}. \tag{2.42}$$

2.2.3 Three dimensions

In three dimensions, the integral to be evaluated is

$$\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz\left(1+(y^2+z^2)\right)^{\frac{1}{2}-s}. \tag{2.43}$$

Changing to polar coordinates, $r^2=y^2+z^2$ with $dy\,dz=dr\,2\pi r$, and using the substitution $u=r^2$, the integral becomes

$$\int_0^{\infty}dr\,2\pi r\left(1+r^2\right)^{\frac{1}{2}-s}=\pi\int_0^{\infty}du\,(1+u)^{\frac{1}{2}-s}=\pi\left[\frac{(1+u)^{\frac{3}{2}-s}}{\frac{3}{2}-s}\right]_0^{\infty}. \tag{2.44}$$

This converges for $s>\frac{3}{2}$, to

$$-\frac{\pi}{s-\frac{3}{2}}. \tag{2.45}$$

Using analytic continuation, we therefore evaluate the integral to be $-\frac{2\pi}{3}$ in the limit $s\to 0$. Also $\zeta(-3)=\frac{1}{120}$. Therefore, the energy as given by (2.36) is

$$\mathcal{E}=\frac{\pi}{8a^3}\cdot\frac{1}{120}\cdot\left(-\frac{2\pi}{3}\right)=-\frac{\pi^2}{1440a^3}. \tag{2.46}$$

2.3 Parallel plates using the argument principle

We now consider two parallel plates separated by a distance a lying perpendicular to the x-direction. We consider the wave equation with wave speed $c(x)$,

$$c^2\nabla^2\varphi-\varphi_{tt}=0, \tag{2.47}$$

where $c(x)$ is constant inside and outside the plates,

$$c(x) = \begin{cases} c_0, & x \in (0, a), \\ c_1, & \text{otherwise.} \end{cases} \tag{2.48}$$

We write the wave equation as

$$c^2 \varphi_{xx} + c^2 \nabla_{\boldsymbol{z}}^2 \varphi - \varphi_{tt} = 0, \tag{2.49}$$

where we view $\varphi = \varphi(x, \boldsymbol{z}, t)$ as a function of the special component x perpendicular to the plates and the parallel components $\boldsymbol{z} = (x_2, \ldots, x_d)$, and the $\boldsymbol{z}$ Laplacian is

$$\nabla_{\boldsymbol{z}}^2 = x_2^2 + \cdots + x_d^2. \tag{2.50}$$

Taking the Fourier transform in time and in each parallel dimension, we get

$$\varphi_{xx} + \left(\frac{\omega^2}{c^2(x)} - k^2 \right) \varphi = 0, \tag{2.51}$$

where $\boldsymbol{k} = (k_2, \ldots, k_N)$ are the wave numbers of each parallel dimension. We can write this as

$$\varphi'' + \lambda^2(x) \varphi = 0, \tag{2.52}$$

where

$$\lambda^2(x) = \frac{\omega^2}{c^2(x)} - k^2. \tag{2.53}$$

The general solutions to (2.52) are

$$\varphi(x) = \begin{cases} A_1 e^{i\lambda_1 x} + B_1 e^{-i\lambda_1 x}, & x < 0 \\ A_2 e^{i\lambda_0 x} + B_2 e^{-i\lambda_0 x}, & 0 < x < a \\ A_3 e^{i\lambda_1 x} + B_3 e^{-i\lambda_1 x}, & a < x \end{cases} \tag{2.54}$$

where

$$\lambda_j = \sqrt{\frac{\omega^2}{c_j^2} - k^2}. \tag{2.55}$$

If $\omega^2/c_1^2 > k^2$ then λ_1 is real. In this case, the terms containing $e^{+\lambda x}$ represent waves moving in positive direction along the x-axis, and the $e^{-\lambda x}$-terms represent waves moving in the negative direction. For $x < 0$, there are no sources that could produce waves moving in positive x-direction,

and this implies that $A_1 = 0$. Similarly, for $x > a$ there are no sources that could produce waves moving in the negative x-direction, so $B_3 = 0$. Next, if $\omega^2/c_0^2 < k^2$ then λ_1 is purely imaginary. In this case, we must have $A_1 = B_3 = 0$ in order to avoid exponential growth in the limit $x \to \pm\infty$. Thus, for all values of λ_1 we have

$$A_1 = B_3 = 0. \tag{2.56}$$

We require that the field and its time derivative are continuous everywhere. In particular, this gives us boundary conditions at $x = 0$ and $x = a$, namely

$$B_1 = A_2 + B_2, \tag{2.57a}$$
$$A_2 e^{i\lambda_0 a} + B_2 e^{-i\lambda_0 a} = A_3 e^{i\lambda_1 a}. \tag{2.57b}$$

A second boundary condition comes from requiring that no energy should be deposited into the boundaries. This can be expressed as saying the normal component of the energy flux

$$\boldsymbol{S} = -c^2 \varphi_t \nabla \varphi \tag{2.58}$$

must be continuous. Using the fact that the normal is in the x-direction and that φ_t is continuous, this means that $-c^2\varphi_x$ is continuous,

$$\begin{aligned} c_1^2 \varphi_x(0^-) &= c_0^2 \varphi_x(0^+), \\ c_0^2 \varphi_x(a^-) &= c_1^2 \varphi_x(a^+). \end{aligned} \tag{2.59}$$

That is,

$$-\lambda_1 c_1^2 B_1 = \lambda_0 c_0^2 A_2 - \lambda_0 c_0^2 B_2, \tag{2.60a}$$
$$\lambda_0 c_0^2 A_2 e^{i\lambda_0 a} - \lambda_0 c_0^2 B_2 e^{-i\lambda_0 a} = \lambda_1 c_1^2 A_3 e^{i\lambda_1 a}. \tag{2.60b}$$

At this point, we drop the subscripts on A_2 and B_2, and write $B_1 = A + B$ and $A_3 e^{i\lambda_1 a} = A e^{i\lambda_0 a} + B e^{-i\lambda_0 a}$. Then (2.57) and (2.60) gives the two boundary conditions

$$-\gamma_1 A - \gamma_1 B = \gamma_0 A - \gamma_0 B, \tag{2.61a}$$
$$\gamma_0 A e^{i\lambda_0 a} - \gamma_0 B e^{-i\lambda_0 a} = \gamma_1 A e^{i\lambda_0 a} + \gamma_1 B e^{-i\lambda_0 a}, \tag{2.61b}$$

where

$$\gamma_j = \lambda_j c_j^2 = c_j \sqrt{\omega^2 - k^2 c_j^2}. \tag{2.62}$$

This can be written on the matrix form

$$M\boldsymbol{b} = \begin{bmatrix} \gamma_1 + \gamma_0 & \gamma_1 - \gamma_0 \\ (\gamma_1 - \gamma_0)e^{i\lambda_0 a} & (\gamma_1 + \gamma_0)e^{-i\lambda_0 a} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0. \tag{2.63}$$

Nontrivial solutions occur when the determinant of the matrix is zero,

$$\det M = (\gamma_1 + \gamma_0)^2 e^{-i\lambda_0 a} - (\gamma_1 - \gamma_0)^2 e^{i\lambda_0 a} = 0. \tag{2.64}$$

Dividing through by γ_1^2 we get the condition

$$(1 + r)^2 e^{-i\lambda_0 a} - (1 - r)^2 e^{i\lambda_0 a} = 0, \tag{2.65}$$

where

$$r = \frac{\gamma_0}{\gamma_1} = \frac{c_0}{c_1}\sqrt{\frac{\omega^2 - k^2 c_0^2}{\omega^2 - k^2 c_1^2}}. \tag{2.66}$$

The possible frequencies ω are those that satisfy this relation. Defining

$$g(\omega, k) = -e^{i\lambda_1 a}\left((1 + r)^2 e^{-i\lambda_0 a} - (1 - r)^2 e^{i\lambda_0 a}\right), \tag{2.67}$$

the possible frequences are ω_n such that $g(\omega_n, k) = 0$. The factor $e^{i\lambda_1 a}$ will become useful later.

The energy of the system is

$$E = \frac{1}{2}\int_{\mathbb{R}^{d-1}} \frac{d\boldsymbol{k}}{(2\pi)^{d-1}} \sum_n \omega_n(k). \tag{2.68}$$

We will evaluate the sum using the argument principle, which says that if h is an analytic function with no poles inside a positively oriented contour C and f is a meromorphic function with no poles or zeroes on C, then

$$\frac{1}{2\pi i}\oint_C dz\, h(z)\frac{d}{dz}\log f(z) = \sum_n m_n h(z_n), \tag{2.69}$$

where z_n are the locations of zeroes and poles of f inside the contour, and m_n are their multiplicities, positive for zeroes and negative for poles.

Choosing $h(z) = z$ and $f(z) = g(z, k)$, having zeroes at $z = \omega_n(k)$, this means that

$$\sum_n \omega_n(k) = \frac{1}{2\pi i}\oint_C dz\, z\frac{d}{dz}\log g(z, k). \tag{2.70}$$

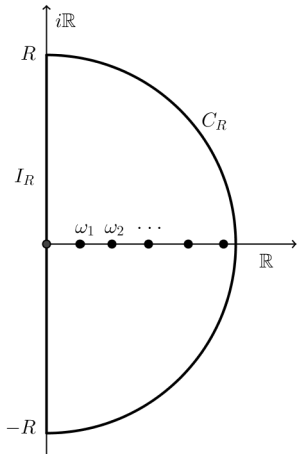


Figure 2.2: Integration curve

We use the contour $C = I_R \cup C_R$, where

$$\begin{aligned} I_R &= \{iy \,|\, y \in [-R, R]\} \\ C_R &= \left\{Re^{i\theta} \,\middle|\, \theta \in [-\pi/2, \pi/2]\right\} \end{aligned} \tag{2.71}$$

as shown in Figure 2.2. Then

$$E = \frac{1}{2i(2\pi)^d} \int_{\mathbb{R}^{d-1}} d\boldsymbol{k} \lim_{R\to\infty} \left[\int_{-iR}^{iR} + \int_{C_R} \right] d\omega\, \omega \frac{d}{d\omega} \log g(\omega, k). \tag{2.72}$$

These integrals are divergent, and in order to regularize them, we will subtract the contribution from high frequencies.

So far we have assumed that no waves will penetrate the plates. In reality, this will not be the case. For high frequencies, the waves will penetrate the plates and act as though they were not there. This means that the speed of light in the medium $c(x)$ is actually also a function of frequency, and that $c(x) \to 1$ as $\omega \to \infty$. In this limit we find that $r \to 1$.

Inserting this in (2.67), the $(1 + r)$ term dominates and we get that the dominating contribution to g for high frequencies is

$$g_\infty(\omega, k) = -(1 + r)^2 e^{i(\lambda_1 - \lambda_0)a}, \tag{2.73}$$

and the corresponding high-frequency contribution to energy is

$$E_\infty = \frac{1}{2i(2\pi)^d} \int_{\mathbb{R}^{d-1}} d\boldsymbol{k} \lim_{R\to\infty} \left[\int_{-iR}^{iR} + \int_{C_R} \right] d\omega\, \omega \frac{d}{d\omega} \log g_\infty(\omega, k). \tag{2.74}$$

We define the regularized energy as the difference between these two,

$$\mathcal{E} = E - E_\infty. \tag{2.75}$$

In the integral over C_R in (2.72), ω is large, meaning the integrand is essentially $\log g_\infty(\omega, k)$. This part cancels completely when subtracting the infinite contribution. What remains is

$$\mathcal{E} = \frac{1}{2i(2\pi)^d} \int_{\mathbb{R}^{d-1}} d\boldsymbol{k} \lim_{R\to\infty} \int_{-iR}^{iR} d\omega\, \omega \frac{d}{d\omega} \log \frac{g(\omega, k)}{g_\infty(\omega, k)}. \tag{2.76}$$

A partial integration and substituting substituting $\omega = iy$ gives

$$\mathcal{E} = \frac{1}{2} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{d-1}} d\boldsymbol{k} \int_{-\infty}^{\infty} dy \log \frac{g(iy, k)}{g_\infty(iy, k)}. \tag{2.77}$$

Here,

$$\frac{g(iy, k)}{g_\infty(iy, k)} = 1 - \frac{(1-r)^2}{(1+r)^2} e^{2i\lambda_0 a}\Big|_{\omega=iy}. \tag{2.78}$$

For our case, consider materials such that $c_0 = 1$ and $c_1 = 0$.[†] Note that inserting this in (2.59) gives von Neumann conditions for φ between the plates. In this limit, $r \to \infty$ and $\lambda_0 \to \sqrt{\omega^2 - k^2}$, and we have

$$\frac{g(iy, k)}{g_\infty(iy, k)} = 1 - e^{2ia\sqrt{(iy)^2 - k^2}} = 1 - e^{-2a\sqrt{y^2+k^2}}. \tag{2.79}$$

Recall that $k^2 = k_2 + \cdots + k_d^2$. Renaming y to k_1, we can write the energy as

$$\mathcal{E} = \frac{1}{2} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\boldsymbol{k} \log\left(1 - e^{-2a\|\boldsymbol{k}\|}\right). \tag{2.80}$$

[†]E.g. vacuum between the plates gives $c_0 = 1$. If the materials are insulating, then waves cannot propagate in them, suggesting $c_1 = 0$.

In polar coordinates, the integral becomes

$$\begin{aligned}\mathcal{E} &= \frac{1}{2}\frac{1}{(2\pi)^d}\int_0^\infty dr\, S_d(r)\log(1-e^{-2ar}) \\ &= \frac{1}{2^d\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})}\int_0^\infty dr\, r^{d-1}\log(1-e^{-2ar})\end{aligned} \tag{2.81}$$

where

$$S_d(r) = \frac{2r^{d-1}\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \tag{2.82}$$

is the surface area of a sphere of radius r in d dimensions. Evaluating this for instance in $d = 2$ dimensions gives the Casimir energy

$$\mathcal{E} = \frac{1}{2\pi}\int_0^\infty dr\, r\log(1-e^{-2ar}) = -\frac{\zeta(3)}{16\pi a^2}. \tag{2.83}$$

2.4 Concentric circles

In this section we will calculate the Casimir energy for two concentric circles in two dimensions. This chapter will rely heavily on the Bessel functions described in Appendix B.

Consider problem (2.1) when the boundaries are two concentric circles in two dimensions, with radii r_1 and r_2 such that $r_1 < r_2$, and normals pointing into the area between the circles. In polar coordinates, (2.1) takes the form

$$\nabla^2\varphi - \varphi_{tt} = 0 \tag{2.84a}$$

$$\varphi_r(r_1,\theta) = \varphi_r(r_2,\theta) = 0. \tag{2.84b}$$

Taking the Fourier transform in the time domain,

$$\nabla^2\varphi + \omega^2\varphi = 0. \tag{2.85}$$

In polar coordinates, the Laplacian is

$$\nabla^2 = \frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial_{\theta\theta} = \nabla_r^2 + \frac{1}{r^2}\nabla_\theta^2. \tag{2.86}$$

where $\nabla_r^2 = \frac{1}{r}\partial_r r \partial_r$ and $\nabla_\theta^2 = \partial_{\theta\theta}$.

Using separation of variables, we look for solutions on the form

$$\varphi(r, \theta) = R(r)\Theta(\theta). \tag{2.87}$$

Inserting this in (2.85) and using the polar form of the Laplacian, we get

$$\Theta\nabla_r^2 R + \frac{1}{r^2} R\nabla_\theta^2\Theta + \omega^2 R\Theta = 0. \tag{2.88}$$

with the boundary condition

$$R'(r_1) = R'(r_2) = 0, \tag{2.89}$$

as well as the periodicity condition,

$$\Theta(\theta + 2\pi) = \Theta(\theta). \tag{2.90}$$

Multiply this equation by $r^2/R\Theta$ to get

$$\frac{r^2\nabla_r^2 R}{R} + \frac{\nabla_\theta^2\Theta}{\Theta} + \omega^2 r^2 = 0. \tag{2.91}$$

Having separated the equation into terms depending only on r or only on θ, we know that these terms must be constant,

$$r^2\left(\omega^2 + \frac{\nabla_r^2 R}{R}\right) = -\frac{\nabla_\theta^2\Theta}{\Theta} = \lambda^2. \tag{2.92}$$

for some constant λ, possibly complex. We now have the set of equations,

$$\begin{aligned} \nabla_r^2 R = \left(\frac{\lambda^2}{r^2} - \omega^2\right) R, \quad & R'(r_1) = R'(r_2) = 0, \\ \nabla_\theta^2\Theta = -\lambda^2\Theta, \qquad & \Theta(\theta + 2\pi) = \Theta(\theta). \end{aligned} \tag{2.93}$$

The angular equation can be written as

$$\Theta'' + \lambda^2\Theta = 0, \tag{2.94}$$

which has general solutions

$$\Theta(\theta) = Ae^{i\lambda(\theta - \varphi)}. \tag{2.95}$$

The periodicity condition $\Theta(\theta + 2\pi) = \Theta(\theta)$ is thus

$$Ae^{i\lambda(\theta-\varphi+2\pi)} = Ae^{i\lambda(\theta-\varphi)}. \tag{2.96}$$

This is equivalent to $e^{2\pi i\lambda} = 1$, which can be satisfied by $\lambda = n$. Let Θ_n and R_n refer to the solutions with that particular n.

The radial equation is

$$\nabla_r^2 R_n = \left(\frac{n^2}{r^2} - \omega^2\right) R_n. \tag{2.97}$$

With

$$\nabla_r^2 R_n = \frac{1}{r}\partial_r r \partial_r R_n = \frac{1}{r}\partial_r (r R_n') = R_n'' + \frac{1}{r}R_n', \tag{2.98}$$

this equation can be written as

$$r^2 R_n'' + r R_n' + \left(\omega^2 r^2 - n^2\right) R_n = 0. \tag{2.99}$$

We recognize this as Bessel's differential equation (B.1) with $x = \omega r$ and order n, so by definition the solutions are

$$R_n(r) = a_n J_n(\omega r) + b_n Y_n(\omega r) \tag{2.100}$$

where J_n and Y_n are the Bessel functions of first kind and second kind respectively. The boundary conditions are

$$R_n'(\omega r_1) = R_n'(\omega r_2) = 0. \tag{2.101}$$

The first condition gives

$$a_n J_n'(\omega r_1) + b_n Y_n'(\omega r_1) = 0 \tag{2.102}$$

which is satisfied if

$$b_n = -a_n \frac{J_n'(\omega r_1)}{Y_n'(\omega r_1)}. \tag{2.103}$$

With this b_n the second boundary condition gives

$$a_n J_n'(\omega r_2) - a_n Y_n'(\omega r_2)\frac{J_n'(\omega r_1)}{Y_n'(\omega r_1)} = 0 \tag{2.104}$$

or

$$J_n'(\omega r_2) Y_n'(\omega r_1) - J_n'(\omega r_1) Y_n'(\omega r_2) = 0. \tag{2.105}$$

This imposes a condition on ω.

Let ω_n^j represent the jth solution to (2.105) for that specific n. The resonance frequencies are precisely these ω_n^j, which we enumerate as ω_n for $n = 1, 2, \ldots$. Then the Casimir energy is given by[†]

$$E = \frac{1}{2} \sum_n \omega_n. \tag{2.106}$$

We will evaluate this sum using the argument principle, which says that for a meromorphic function $f(z)$ with no poles or zeroes on the contour C and a function $h(z)$ that is analytic inside C,

$$\frac{1}{2\pi i} \oint_C dz\, h(z) \frac{d}{dz} \log f(z) = \sum_n m_n h(z_n), \tag{2.107}$$

where z_n are the locations of zeroes and poles of $f(z)$ inside the contour and m_n are their multiplicities, viewing poles as zeroes with negative multiplicity. The function

$$f(\omega) = \prod_n \left(J_n'(\omega r_2) Y_n'(\omega r_1) - J_n'(\omega r_1) Y_n'(\omega r_2) \right) \tag{2.108}$$

satisfies the condition $f(\omega_n = 0$ for all n, and $h(z) = z$ means the right-hand side of (2.107) is exactly the sum in the expression for the Casimir energy. For the contour C, select a semicircle centred at the origin with radius R in the right half plane, and its diameter. We must move in a small semicircle of radius ε around the origin, since this is a branch point of $\log f(z)$. The contour is shown in Figure 2.3, and can be described as

$$C = C_R \cup I_{\varepsilon,R} \cup C_\varepsilon \cup I_{-\varepsilon,-R}, \tag{2.109}$$

where

$$\begin{aligned} I_{a,b} &= \{ iy \,|\, y \in [a,b] \} \\ C_r &= \left\{ re^{i\theta} \,\middle|\, \theta \in [-\pi/2, \pi/2] \right\}, \end{aligned} \tag{2.110}$$

the Casimir energy can be written as

$$E = \frac{1}{4\pi i} \lim_{R\to\infty} \oint_C dz\, z \frac{d}{dz} \log f(z). \tag{2.111}$$

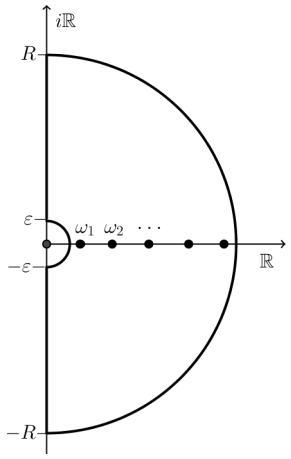


Figure 2.3: The contour C

This integral diverges, and we regularize it by subtracting the high frequency contribution.

We need the properties

$$\begin{aligned} J_n'(ix) &= i^{n-1} I_n'(x), \\ Y_n'(ix) &= i^n I_n'(x) - \frac{2}{\pi} i^{-(n+1)} K_n'(x), \end{aligned} \tag{2.112}$$

which can be derived from (B.8). It follows that

$$\begin{aligned} J_n'(ia) J_n'(ia) &= i^{n-1} I_n'(a) \left(i^n I_n'(b) - \frac{2}{\pi} i^{-n-1} K_n'(b) \right) \\ &= i^{2n-1} I_n'(a) I_n'(b) - \frac{2}{\pi} i^{-2} I_n'(a) K_n'(b) \\ &= \frac{2}{\pi} I_n'(a) K_n'(b) - i(-1)^n I_n'(a) I_n'(b). \end{aligned} \tag{2.113}$$

[†]Finding all the ω_n explicitly is hard, which is why we use the argument principle instead of doing it in a way similar to what was described in Section 2.1.

Evaluating $f(z)$ at $z = iy$, we get

$$\begin{aligned} f(iy) &= \prod_n \left(J_n'(iyr_2)Y_n'(iyr_1) - J_n'(iyr_1)Y_n'(iyr_2)\right) \\ &= \prod_n \left(\frac{2}{\pi} I_n'(yr_2)K_n'(yr_1) - i(-1)^n I_n'(yr_1)I_n'(yr_2)\right. \\ &\qquad \left. -\frac{2}{\pi} I_n'(yr_1)K_n'(yr_2) + i(-1)^n I_n'(yr_2)I_n'(yr_1)\right) \\ &= \prod_n \frac{2}{\pi}\left(I_n'(yr_2)K_n'(yr_1) - I_n'(yr_1)K_n'(yr_2)\right). \end{aligned} \tag{2.114}$$

According to (B.14), in the limit of large x, the asymptotic from of the product $I_n'(ax)K_n'(bx)$ is

$$I_n'(ax)K_n'(bx) \sim -\frac{1}{2x} e^{(a-b)x}, \tag{2.115}$$

which will be dominated by I_n' if $a > b$ and in this case grow exponentially, and it will be dominated by K_n' and vanish exponentially if $a < b$. Since $r_2 > r_1$, this means that for large values of y,

$$f(iy) \sim f_\infty(iy) = \prod_n \frac{2}{\pi} I_n'(yr_2)K_n'(yr_1). \tag{2.116}$$

By using an analytic continuation, replace $y = -iz$ to get

$$f_\infty(z) = \prod_n \frac{2}{\pi} I_n'(-izr_2)K_n'(-izr_1). \tag{2.117}$$

The argument leading to (2.116) only works when y is real. Therefore, applying analytic continuation like this should be regarded as a regularization that is not strictly rigorous. Taking the limit of the Hankel function $H_\alpha^{(1)}(x)$ as α tends to some positive integer n, it can be shown that

$$H_n^{(1)}(x) = 2J_n(x). \tag{2.118}$$

This together with the relation

$$K_\alpha'(x) = -\frac{\pi}{2} i^\alpha H_\alpha^{(1)\prime}(ix), \qquad -\pi < \arg x \leq \frac{\pi}{2} \tag{2.119}$$

from (B.7), and with (2.112), allows us to rewrite

$$\begin{aligned} f_\infty(z) &= \prod_n \frac{2}{\pi} I_n'(-izr_2) K_n'(-izr_1) \\ &= \prod_n \frac{2}{\pi} \left(i^{1-n} J_n'(-zr_2) \right) \left(-\frac{\pi}{2} i^n H_n^{(1)\prime}(-zr_1) \right) \\ &= \prod_n -2i J_n'(zr_1) J_n'(zr_2). \end{aligned} \tag{2.120}$$

We define the infinite contribution of the energy E_∞ as the energy acquired when substituting f_∞ for f in (2.111), then regularize the energy by subtracting the infinite part of the energy:

$$\mathcal{E} = E - E_\infty = \frac{1}{4\pi i} \oint_C dz\, z \frac{d}{dz} \log \frac{f(z)}{f_\infty(z)}. \tag{2.121}$$

We shall split this integral into four pieces, one over each segment. Using the fact that $f(-z) = f(z)$ and a partial integration, we can thus write the energy as

$$\begin{aligned} \mathcal{E} = &- \frac{1}{2\pi} \int_0^\infty dy\, \log \frac{f(iy)}{f_\infty(iy)} \\ &- \frac{1}{4\pi} \lim_{\varepsilon \to 0} \int_{-\pi/2}^{\pi/2} d\theta\, \varepsilon e^{i\theta} \log \frac{f(\varepsilon e^{i\theta})}{f_\infty(\varepsilon e^{i\theta})} \\ &+ \frac{1}{4\pi} \lim_{R \to \infty} \int_{-\pi/2}^{\pi/2} d\theta\, R e^{i\theta} \log \frac{f(R e^{i\theta})}{f_\infty(R e^{i\theta})}. \end{aligned} \tag{2.122}$$

We will now treat each of these integrals one by one.

The first integral is

$$\begin{aligned} \int_0^\infty dy\, \log \frac{f(iy)}{f_\infty(iy)} &= \int_0^\infty dy\, \log \prod_n \frac{I_n'(yr_2) K_n'(yr_1) - I_n'(yr_1) K_n'(yr_2)}{I_n'(yr_2) K_n'(yr_1)} \\ &= \sum_n \int_0^\infty dy\, \log \left(1 - \frac{I_n'(yr_1) K_n'(yr_2)}{I_n'(yr_2) K_n'(yr_1)} \right). \end{aligned} \tag{2.123}$$

Using the asymptotic forms of I_n and K_n in (B.12) we see that

$$\lim_{y\to 0} \frac{I_n'(yr_1)K_n'(yr_2)}{I_n'(yr_2)K_n'(yr_1)} = \left(\frac{r_1}{r_2}\right)^{2n}, \tag{2.124}$$

and (B.14) indicates that this term drops off exponentially as $y \to \infty$. With $\log(1-x) = -x + \mathcal{O}(x^2)$, this indicates that each integral converges and is of order $(r_1/r_2)^{2n}$, and thus the sum converges at a rate comparable to a geometric sum. Therefore, when evaluating this sum numerically, we choose a cutoff after n terms and assume the error will be of order

$$e \sim \frac{(r_1/r_2)^{2n}}{1 - r_1/r_2}. \tag{2.125}$$

The second integral is

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{-\pi/2}^{\pi/2} d\theta\, \varepsilon e^{i\theta} \log \frac{f(\varepsilon e^{i\theta})}{f_\infty(\varepsilon e^{i\theta})} \\
&= \lim_{\varepsilon\to 0} \sum_n \int_{-\pi/2}^{\pi/2} d\theta\, \varepsilon e^{i\theta} \log \left(\frac{J_n'(zr_2)Y_n'(zr_1) - J_n'(zr_1)Y_n'(zr_2)}{-2iJ_n'(zr_1)J_n'(zr_2)} \right) \\
&= \lim_{\varepsilon\to 0} \sum_n \int_{-\pi/2}^{\pi/2} d\theta\, \varepsilon e^{i\theta} \log \left(\frac{iY_n'(zr_1)}{2J_n'(zr_1)} - \frac{iY_n'(zr_2)}{2J_n'(zr_2)} \right) \\
&= \lim_{\varepsilon\to 0} \sum_n \int_{-\pi/2}^{\pi/2} d\theta\, \varepsilon e^{i\theta} \operatorname{Log} \left| \frac{Y_n'(\varepsilon r_1 e^{i\theta})}{2J_n'(\varepsilon r_1 e^{i\theta})} - \frac{Y_n'(\varepsilon r_2 e^{i\theta})}{2J_n'(\varepsilon r_2 e^{i\theta})} \right|,
\end{aligned} \tag{2.126}$$

where Log indicates the function on the principal branch. We also get some constant terms from log, but they vanish when $\varepsilon \to 0$. From (B.12),

$$\begin{aligned}
J_0'(x) &= -\frac{1}{2}x, \qquad Y_0'(x) = \frac{2}{\pi}\frac{1}{x}, \\
J_n'(x) &= \frac{nx^{n-1}}{2^n \Pi(n)}, \quad Y_n'(x) = -\frac{2^n \Pi(n)}{\pi x^{n+1}}.
\end{aligned} \tag{2.127}$$

Then for $n = 0$,

$$\left| \frac{Y_0'(\varepsilon r_1 e^{i\theta})}{2J_0'(\varepsilon r_1 e^{i\theta})} - \frac{Y_0'(\varepsilon r_2 e^{i\theta})}{2J_0'(\varepsilon r_2 e^{i\theta})} \right| = \left| \frac{\frac{2}{\pi}\frac{1}{\varepsilon r_1 e^{i\theta}}}{\varepsilon r_1 e^{i\theta}} - \frac{\frac{2}{\pi}\frac{1}{\varepsilon r_2 e^{i\theta}}}{\varepsilon r_2 e^{i\theta}} \right| = \frac{2}{\varepsilon^2 \pi} \left(\frac{1}{r_1^2} - \frac{1}{r_2^2} \right). \tag{2.128}$$

and for $n \neq 0$,

$$\begin{aligned} &\left| \frac{Y_n'(\varepsilon r_1 e^{i\theta})}{2J_n'(\varepsilon r_1 e^{i\theta})} - \frac{Y_n'(\varepsilon r_2 e^{i\theta})}{2J_n'(\varepsilon r_2 e^{i\theta})} \right| \\ &= \left| -\frac{2^n \Pi(n) \cdot 2^n \Pi(n)}{\pi(\varepsilon r_1 e^{i\theta})^{n+1} \cdot n(\varepsilon r_1 e^{i\theta})^{n-1}} + \frac{2^n \Pi(n) \cdot 2^n \Pi(n)}{\pi(\varepsilon r_2 e^{i\theta})^{n+1} \cdot n(\varepsilon r_2 e^{i\theta})^{n-1}} \right| \\ &= \frac{2^{2n} \Pi^2(n)}{\pi n \varepsilon^{2n}} \left(\frac{1}{r_1^{2n}} - \frac{1}{r_2^{2n}} \right). \end{aligned} \tag{2.129}$$

In both of these cases, there is a power relation with ε. Hence, when we take the logarithm we will get integrands proportional to ε or $\varepsilon \log \varepsilon$. As $\varepsilon \to 0$, these vanish, so this integral term is zero.

Note however that as n increases, the terms grow without bounds. This means the sum in (2.126) does not actually converge unless we take the limit first. Therefore, summing the limits instead of taking the limit of the sum should be regarded as a regularization step. Then we have

$$-\frac{1}{4\pi} \lim_{\varepsilon \to 0} \int_{-\pi/2}^{\pi/2} d\theta \, \varepsilon e^{i\theta} \log \frac{f(\varepsilon e^{i\theta})}{f_\infty(\varepsilon e^{i\theta})} = 0. \tag{2.130}$$

The final integral is

$$\lim_{R \to \infty} \int_{-\pi/2}^{\pi/2} d\theta R e^{i\theta} \log \frac{f(R e^{i\theta})}{f_\infty(R e^{i\theta})}. \tag{2.131}$$

Unfortunately, when we try to evaluate this integral, we will find that it diverges in the limit! This mean we would need another regularization step. However, suppose we were to find it to be zero. Then the energy would be

$$\mathcal{E} = -\frac{1}{2\pi} \sum_n \int_0^\infty dy \, \log \left(1 - \frac{I_n'(yr_1) K_n'(yr_2)}{I_n'(yr_2) K_n'(yr_1)} \right). \tag{2.132}$$

Kilen similarly removed an infinite term and showed that the remaining term matched his numerical solution [14]. With this knowledge, we feel reassured that (2.132) gives the correct energy, despite not having a satisfactorily rigorous explanation.

2.5 The relation between energy and pressure

This section describes how to relate the Casimir energy to the force. The force on a small surface element dA on one of the objects is given by

$$d\boldsymbol{F} = -\boldsymbol{n}p\,dA, \tag{2.133}$$

where $\boldsymbol{n}$ is the outwards-pointing normal and p is the pressure at that point. By displacing this surface element by a distance $d\boldsymbol{r}$, we perform a work equal to

$$dW = d\boldsymbol{F} \cdot d\boldsymbol{r}. \tag{2.134}$$

Integrating the contributions from all surface elements yields

$$dE = -\int_Q d\boldsymbol{A} \cdot p\,d\boldsymbol{r}. \tag{2.135}$$

On the other hand, the energy of a configuration E depends on a set of ν variables (for example the radii of two concentric circles),

$$E = E(q_1, \ldots, q_\nu). \tag{2.136}$$

Let

$$\gamma(s) = \gamma(q_1(s), \ldots, q_\nu(s)) \tag{2.137}$$

be a curve in parameter space. Then the differential of E in terms of s is

$$dE(\gamma(s)) = \nabla E \cdot \gamma'(s)\,ds. \tag{2.138}$$

Thus the energy is related to the pressure according to

$$\nabla E \cdot \gamma'(s) = -\int_Q d\boldsymbol{A} \cdot p\,\frac{d\boldsymbol{r}}{ds}. \tag{2.139}$$

The factor $\frac{d\boldsymbol{r}}{ds}$ can be thought of as the velocity of a point on Q as the configuration changes along γ.

Example: Parallel Plates

Consider the case of two parallel plates in two dimensions. The positions x_1 and x_2 of the plates are the parameters of our configuration. As we vary the position of the leftmost plate, select $\gamma(s) = (s, a)$ to be our curve through parameter space. Then the change in energy in terms of s is

$$\frac{dE}{ds} = \nabla E \cdot \gamma'(s) = (\partial_{x_1} E, \partial_{x_2} E) \cdot (1, 0) = \frac{\partial E}{\partial x_1}. \tag{2.140}$$

For a given s, we parametrise the left curve as

$$\boldsymbol{r}_1(s, y) = (s, y). \tag{2.141}$$

The derivative is

$$\frac{d\boldsymbol{r}_1}{ds} = (1, 0). \tag{2.142}$$

We parametrise the right curve as $\boldsymbol{r}_2(s, y) = (a, y)$, which does not vary with respect to s. With constant pressure, we then get

$$-\int_Q d\boldsymbol{A} \cdot p \frac{d\boldsymbol{r}}{ds} = -\int_{Q_1} dA\, p = -Lp. \tag{2.143}$$

Equating (2.140) and (2.143) gives the pressure in terms of energy per unit length,

$$p_1 = \frac{\partial E}{\partial x_1}. \tag{2.144}$$

Proceeding similarly for p_2 we find $p_2 = -\partial_{x_2} E$. Since $a = x_2 - x_1$, this gives

$$p_1 = p_2 = -\frac{\partial E}{\partial a}. \tag{2.145}$$

The exact Casimir energy for two infinitely long parallel plates in two dimensions was found to be

$$\mathcal{E} = -\frac{\zeta(3)}{16\pi a^2}, \tag{2.146}$$

so the pressure is

$$p_1 = p_2 = -\frac{\zeta(3)}{8\pi a^3}. \tag{2.147}$$

The same procedure can be used to show that (2.145) is valid for any number of dimensions. for example for one dimension,

$$p_{(1\mathrm{D})} = -\frac{\partial}{\partial a}\left(-\frac{\pi}{24a}\right) = -\frac{\pi}{24a^2}. \tag{2.148}$$

Example: Concentric Circles

For two concentric circles, the with radii r_1 and r_2 are the parameters of our configuration. Treating the inner circle first, we choose the curve in parameter space $\gamma(s) = (s, R)$, where R is constant. Then

$$\frac{dE}{ds} = \nabla E \cdot \gamma'(s) = \frac{\partial E}{\partial r_1}. \tag{2.149}$$

Parametrise the inner circle as

$$\boldsymbol{r}_1(s,t) = (s\cos t, s\sin t) = s\boldsymbol{n}(t). \tag{2.150}$$

The derivative of $\boldsymbol{r}_1$ with respect to s is $\boldsymbol{n}$. We also have $\boldsymbol{r}_2(s,t) = R\boldsymbol{n}(t)$, which does not change as s changes. Thus,

$$-\int_Q d\boldsymbol{A} \cdot p\frac{d\boldsymbol{r}}{ds} = -\int_{Q_1} d\boldsymbol{A} \cdot \boldsymbol{n}p = -2\pi r_1 p. \tag{2.151}$$

We have used the fact that pressure must be constant due to rotational symmetry. Combining (2.149) and (2.151) gives

$$p_1 = -\frac{1}{2\pi r_1}\frac{\partial E}{\partial r_1}. \tag{2.152}$$

A similar process for the outer circle gives

$$p_2 = \frac{1}{2\pi r_2}\frac{\partial E}{\partial r_2}. \tag{2.153}$$

From mode expansion (2.132), we found that

$$\mathcal{E} = -\frac{1}{2\pi}\sum_k \int_0^\infty dy\, \log\left(1 - \frac{I_k'(yr_1)K_k'(yr_2)}{I_k'(yr_2)K_k'(yr_1)}\right), \tag{2.154}$$

so for example, the pressure on Q_1 is

$$p_1 = -\frac{1}{4\pi^2 r_1}\sum_k \int_0^\infty dy\, \frac{\partial}{\partial r_1}\log\left(1 - \frac{I_k'(yr_1)K_k'(yr_2)}{I_k'(yr_2)K_k'(yr_1)}\right). \tag{2.155}$$

Chapter 3

The functional integral method

In this section, we present the functional integral method, as applied to objects with von Neumann boundary conditions. This is based on the work by T. Emig et. al., in particular on [10], and on the procedure used by Kilen [14] and Mikalsen [20].

Section 3.1 will give a preliminary introduction to the functional integral. Here we discuss the notion of a functional integral, and techniques for how to evaluate them. Then in Section 3.2, we express the Casimir energy as a functional integral. Section 3.3, Section 3.4 and Section 3.5 will rewrite this as a Gaussian integral which can be evaluated, each section concerning different aspects of the integral. The evaluation itself is discussed in Section 3.6.

In Section 3.4 we will find that there is a hole in our argument. This lapse is also found in [10]. We will present some heuristic points to justify the conclusion, but we will not give a rigorous proof. Finally, in Section 3.7 we will present a way to avoid this problem, but at a severe increase in computation cost.

3.1 Functional Integrals

Before we begin, let us take a moment to look at what the end result might look like. We will express the Casimir force in terms of a functional integral,

$$\int D\varphi\, F[\varphi], \tag{3.1}$$

where F is a functional, mapping a function to a number, e.g. the integral functional

$$I[\varphi] = \int_{-\infty}^{\infty} dx\, \varphi(x). \tag{3.2}$$

The functional integral means an integral over a domain of all possible functions, instead of a regular integral over numbers. That is, if $\varphi : \mathbb{R} \to \mathbb{R}$, the functional differential $D\varphi$ can be thought of as

$$D\varphi = \prod_x d\varphi(x), \tag{3.3}$$

meaning a multiple integral, each integral with respect to the function value $\varphi(x)$ at the point x. This means an infinite number of integrals - and in fact, even an uncountably infinite number. It is certainly not immediately obvious how to evaluate such an integral.

One way to make sense of the functional integral is by decomposing φ as

$$\varphi(x) = \sum_n \varphi_n \psi_n(x), \tag{3.4}$$

where $\{\psi_n\}$ is a complete set of functions, and interpreting the integral as

$$\int D\varphi\, F[\varphi] = \int d\varphi_1 \int d\varphi_2 \cdots \hat{F}(\varphi_1, \varphi_2, \ldots), \tag{3.5}$$

where $\hat{F}$ is now a normal function of an infinite number of variables. This is still an infinite number of integrals, but at least now it is countable. This kind of integrals were described by P. J. Daniell [8] who showed how it can be evaluated for certain functions $\hat{F}$. For example, if we integrate over the functions defined on $[0, 1]$, we can use the finite Fourier transform to decompose them in the basis

$$\psi_n(x) = e^{2\pi i n x}. \tag{3.6}$$

The integral functional becomes

$$I[\varphi] = \int_0^1 dx\, \varphi(x) = \sum_n \varphi_n \int_0^1 dx\, e^{2\pi i n x} = \varphi_0, \tag{3.7}$$

so in fact, $\hat{I}(\varphi_0, \varphi_1, \ldots) = \hat{I}(\varphi_0)$ is actually just a function of the first variable φ_0. Then the functional integral of I is

$$\int D\varphi\, I[\varphi] = \int d\varphi_0\, \varphi_0 \cdot \int d\varphi_1 \int d\varphi_2 \cdots. \tag{3.8}$$

If we have two functionals F and G that are functions of only the first variable, $\hat{F} = \hat{F}(\varphi_0)$ and $\hat{G} = \hat{G}(\varphi_0)$, then the trailing integrals cancel and the ratio between the two functional integrals can be understood as

$$\frac{\int D\varphi\, F[\varphi]}{\int D\varphi\, G[\varphi]} = \frac{\int d\varphi_0\, \hat{F}(\varphi_0)}{\int d\varphi_0\, \hat{G}(\varphi_0)}. \tag{3.9}$$

When calculating the Casimir force, a common regularization is to subtract the contribution of high frequencies. Mathematically, if we have the basis (3.6), this interpretation follows from setting $\varphi_m = 0$ for all m larger than some N, and the regularizations eventually yield integrals on the form

$$\int d\varphi_0 \cdots \int d\varphi_N\, \hat{F}(\varphi_0, \cdots, \varphi_N). \tag{3.10}$$

This process can be regarded as a form of discretization, and is discussed again in Section 3.6.

In our case, the kind of functionals we will encounter yield Gaussian integrals. We will now give a short introduction to Gaussian integrals and derive the important result that we will need later. Start by considering the one-dimensional Gaussian integral

$$I = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}\lambda x^2}, \qquad \lambda > 0. \tag{3.11}$$

By squaring and introducing polar coordinates, we can write this as

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}\lambda x^2} \int_{-\infty}^{\infty} dy\, e^{-\frac{1}{2}\lambda y^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy\, e^{-\frac{1}{2}\lambda(x^2+y^2)} \\ &= \int_0^{2\pi} d\theta \int_0^{\infty} dr\, r e^{-\frac{1}{2}\lambda r^2} = 2\pi \int_0^{\infty} \frac{du}{\lambda} e^{-u} = \frac{2\pi}{\lambda}, \end{aligned} \tag{3.12}$$

which implies that

$$I = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}\lambda x^2} = \sqrt{\frac{2\pi}{\lambda}}. \tag{3.13}$$

Next we move on to complex arguments. Observe that

$$e^{-\frac{1}{2}(x^2+y^2)} = e^{-\frac{1}{2}(x+iy)(x-iy)}. \tag{3.14}$$

Let

$$z = \frac{1}{\sqrt{2}}(x+iy), \qquad z^* = \frac{1}{\sqrt{2}}(x-iy). \tag{3.15}$$

The differentials become $dx\, dy = -i dz\, dz^*$, and we see from (3.12) that

$$\int_{\mathbb{C}} \frac{dz\, dz^*}{2\pi i} e^{-\lambda|z|^2} = \int_{\mathbb{C}} \frac{dz\, dz^*}{2\pi i} e^{-\lambda z z^*} = \frac{1}{\lambda}. \tag{3.16}$$

Now let $D = (d_{ij})$ be an $n \times n$ diagonal matrix with elements λ_j along the diagonal, such that D is positive definite, i.e. all $\lambda_j > 0$. Then

$$\langle \boldsymbol{z}, D\boldsymbol{z} \rangle = \sum_{i,j} z_i^* d_{ij} z_j = \sum_j \lambda_j |z_j|^2, \tag{3.17}$$

and the determinant of D is given by

$$\det D = \prod_j \lambda_j. \tag{3.18}$$

Abbreviating

$$[d\boldsymbol{z}\, d\boldsymbol{z}^*] = \prod_j \frac{dz_j\, dz_j^*}{2\pi i} \tag{3.19}$$

and using these results, we can solve the Gaussian integral

$$\begin{aligned} \int_{\mathbb{C}^n} [d\boldsymbol{z}\, d\boldsymbol{z}^*]\, e^{-\langle \boldsymbol{z}, D\boldsymbol{z}\rangle} &= \int_{\mathbb{C}^n} [d\boldsymbol{z}\, d\boldsymbol{z}^*]\, e^{-\sum_j \lambda_j |z_j|^2} \\ &= \prod_j \int_{\mathbb{C}^n} \frac{dz_j\, dz_j^*}{(2\pi i)^n} e^{-\lambda_j |z_j|^2} \\ &= \prod_j \frac{1}{\lambda_j} = \frac{1}{\det D}. \end{aligned} \tag{3.20}$$

Let A be a positive definite, Hermitian matrix. There exists a unitary matrix U such that

$$U^*AU = D \tag{3.21}$$

for some diagonal matrix D consisting of the eigenvalues of A. Since A is positive definite and Hermitian, these eigenvalues are real and positive. Introduce the change of variables $\boldsymbol{z} = U\boldsymbol{z}'$. Then

$$\begin{aligned}\int_{\mathbb{C}^n} [d\boldsymbol{z}\, d\boldsymbol{z}^*]\, e^{-\langle \boldsymbol{z}, A\boldsymbol{z}\rangle} &= \int_{\mathbb{C}^n} [d(U\boldsymbol{z}')\, d(U\boldsymbol{z}')^*]\, e^{-\langle U\boldsymbol{z}', AU\boldsymbol{z}'\rangle} \\ &= \int_{\mathbb{C}^n} \left[d\boldsymbol{z}'\, d\boldsymbol{z}'^*\right] (\det U \det U^*) e^{-\langle \boldsymbol{z}', U^*AU\boldsymbol{z}'\rangle} \\ &= \int_{\mathbb{C}^n} \left[d\boldsymbol{z}'\, d\boldsymbol{z}'^*\right] e^{-\langle \boldsymbol{z}', D\boldsymbol{z}'\rangle} = \frac{1}{\det D}.\end{aligned} \tag{3.22}$$

Using the property that

$$\det D = \det(U^*AU) = \det A \cdot \det(UU^*) = \det A, \tag{3.23}$$

this means that

$$\int_{\mathbb{C}^n} \left(\prod_j \frac{dz_j\, dz_j^*}{2\pi i} \right) e^{-\langle \boldsymbol{z}, A\boldsymbol{z}\rangle} = \frac{1}{\det A} \tag{3.24}$$

for any positive definite, Hermitian matrix A.

3.2 Energy as a Functional Integral

Consider the massless scalar field $\varphi(\boldsymbol{x}, t)$, described by the Langrangian

$$\mathcal{L}\varphi = \frac{1}{2}\varphi_t^2 - \frac{1}{2}(\nabla\varphi)^2. \tag{3.25}$$

Performing a canonical quantization on φ yields a quantum operator $\hat{\varphi}$ that is related to the classical field via the eigenstate equation

$$\hat{\varphi}(\boldsymbol{x})\,|\varphi\rangle = \varphi(\boldsymbol{x})\,|\varphi\rangle\,. \tag{3.26}$$

In the Heisenberg picture, time dependency is introduced through the equation of motion,

$$-i\hbar\partial_t\hat{\varphi} = \left[\hat{H}, \hat{\varphi}\right], \tag{3.27}$$

and the time dependence of the eigenstates is given by

$$|\varphi, t\rangle = e^{it\hat{H}} |\varphi\rangle . \tag{3.28}$$

The amplitude of a transition between two states can be written as

$$\langle \varphi_1, t_1 | \varphi_0, t_0 \rangle = \left\langle \varphi_1 \middle| e^{-i(t_1 - t_0)\hat{H}} \middle| \varphi_0 \right\rangle . \tag{3.29}$$

This transition amplitude can be written as a functional integral (see [12]) according to

$$\left\langle \varphi_1 \middle| e^{-i(t_1 - t_0)\hat{H}} \middle| \varphi_0 \right\rangle = \int D\varphi \, e^{iS[\varphi]}, \tag{3.30}$$

where the integration is over all fields satisfying the boundary conditions $\varphi(t_0) = \varphi_0$ and $\varphi(t_1) = \varphi_1$, and S is the action,

$$S[\varphi] = \int_{t_0}^{t_1} dt \int_{\mathbb{R}^d} d\boldsymbol{x} \, \mathcal{L}\varphi(\boldsymbol{x}, t). \tag{3.31}$$

We will study the case with the conditions $\varphi(x, 0) = \varphi(x, T) = 0$. That is, we are interested in the vacuum to vacuum transition after a time period $T = t_1 - t_0$. This constraint means that the fields will be T-periodic. The amplitude for this transition is then given by the partition function

$$Z_Q(S, T) = \int D\varphi_{C,T} \, e^{iS[\varphi]}. \tag{3.32}$$

The integration variable $D\varphi_{C,T}$ indicates that the integration is over all T-periodic fields that are subject to boundary conditions C, given by

$$C: \quad \partial_n \varphi \Big|_{\mathcal{Q}} = 0, \tag{3.33}$$

on the space-time surface $\mathcal{Q} = Q \times [0, T]$.

Let $|\alpha\rangle$ be a complete set of energy eigenstates, i.e. $\hat{H} |\alpha\rangle = E_\alpha |\alpha\rangle$ with $\langle \alpha | \beta \rangle = \delta_{\alpha\beta}$, and let $|0\rangle$ be the state of the vacuum. Then the vacuum to

vacuum transition can be written as

$$\begin{aligned} Z_Q(S,T) &= \left\langle 0 \middle| e^{-i(t'-t)\hat{H}} \middle| 0 \right\rangle \\ &= \langle 0| \sum_\alpha |\alpha\rangle \langle\alpha| \, e^{-iT\hat{H}} \sum_\beta |\beta\rangle \langle\beta|0\rangle \\ &= \sum_{\alpha,\beta} \langle 0|\alpha\rangle \langle\beta|0\rangle \left\langle \alpha \middle| e^{-iT\hat{H}} \middle| \beta \right\rangle \\ &= \sum_{\alpha,\beta} \langle 0|\alpha\rangle \langle 0|\beta\rangle^* \langle\alpha|\beta\rangle \, e^{-iTE_\beta} \\ &= \sum_\alpha |\langle 0|\alpha\rangle|^2 \, e^{-iTE_\alpha}. \end{aligned} \tag{3.34}$$

In order to obtain a convergent series, we perform a Wick rotation, $T = -is$ where s is real. Then

$$Z_Q(S,T) = \sum_\alpha |\langle 0|\alpha\rangle|^2 \, e^{-sE_\alpha}. \tag{3.35}$$

For large s, we can approximate

$$Z_Q(S,T) \approx |\langle 0|\alpha_0\rangle|^2 \, e^{-sE_0}, \tag{3.36}$$

where E_0 is the lowest energy state and $|\alpha_0\rangle$ is the corresponding eigenvector. We can solve this for E_0:

$$E_0 \approx \frac{2}{s} \log |\langle 0|\alpha_0\rangle| - \frac{1}{s} \log Z_Q(S,-is). \tag{3.37}$$

In the limit $s \to \infty$, the second term dominates and we have

$$E_0 = -\lim_{s\to\infty} \frac{1}{s} \log Z_Q(S,-is). \tag{3.38}$$

This quantity is infinite and must be regularized.

Let E_∞ be the energy and Z_∞ be the partition function obtained when the objects have been moved to infinite separation. It follows the same relation,

$$E_\infty = -\lim_{s\to\infty} \frac{1}{s} \log Z_\infty(-is). \tag{3.39}$$

This can be viewed as the ground state energy of the vacuum when there are no objects, the free vacuum energy. Introducing objects creates a fluctuation

in this energy, and this is what gives rise to the Casimir forces. Since the forces are related to the gradient of the energy and not its actual value, we argue that the energy E_0 can be regularized by subtracting the free vacuum energy E_∞,

$$\mathcal{E} = E_0 - E_\infty = -\lim_{s\to\infty} \frac{1}{s} \log \frac{Z_Q(S,-is)}{Z_\infty(-is)}. \tag{3.40}$$

This quantity will turn out to be finite.

In summary, we have an expression for $\mathcal{E}$ in terms of Z_Q and Z_∞. We must now derive expressions for $Z_Q(S,T)$ and $Z_\infty(T)$. Finding Z means evaluating the integral

$$Z_Q(S,T) = \int D\varphi_{C,T}\; e^{iS[\varphi]}, \tag{3.41}$$

where the integral is over all curves φ that satisfy $\partial_n\varphi|_{\mathcal{Q}} = 0$ and that are T-periodic. We will approach this integral using the methods described in Section 3.1. First, in Section 3.3, we modify this integral to implement the conditions in the integrand using an analogy for the Dirac δ function and Fourier transforms, so that the integration will be unconstrained over all functions. This will introduce a new function ρ and an integral $\int D\rho$. In Section 3.4 we resolve the integral over $D\varphi$, and in Section 3.5 we rewrite the integral over $D\rho$ as a Gaussian integral. Finally we discretize it in Section 3.6 and assign to it a value.

3.3 Implementing Conditions

The delta functional is the functional analogy to the Dirac delta function. It has the property

$$\int D\varphi\, \delta[A\varphi|_C] F[\varphi] = \int D\psi\, F[\psi], \tag{3.42}$$

where the integral on the right hand side is over all functions that satisfy

$$A\psi|_C = 0. \tag{3.43}$$

In Appendix A we argue that the delta functional can be defined as

$$\delta[A\varphi|_C] = \int D\rho\, e^{i\int_C d\boldsymbol{x}\, \rho(\boldsymbol{x}) A\varphi(\boldsymbol{x})}. \tag{3.44}$$

Using this, we can implement our boundary condition

$$\partial_{\boldsymbol{n}}\varphi\Big|_{\mathcal{Q}} = 0 \tag{3.45}$$

in terms of the delta functional $\delta[\partial_{\boldsymbol{n}}\varphi|_{\mathcal{Q}}]$. That is,

$$Z = \int D\varphi_{C,T}\, e^{iS[\varphi]} = \int D\varphi_T\, D\rho\, e^{i\int_Q dA\int_0^T dt\,\rho(\boldsymbol{x},t)\partial_{\boldsymbol{n}}\varphi(\boldsymbol{x},t)} e^{iS[\varphi]}. \tag{3.46}$$

Since the integral only concerns $\rho(\boldsymbol{x},t)$ for $t\in[0,T]$, we can assume ρ is periodic with period T. Then we can expand φ and ρ into Fourier series,

$$\begin{aligned}\varphi_T(\boldsymbol{x},t) &= \sum_{n=-\infty}^{\infty} \varphi_n(\boldsymbol{x}) e^{2\pi i n t/T}\\ \rho(\boldsymbol{x},t) &= \sum_{n=-\infty}^{\infty} \rho_n(\boldsymbol{x}) e^{2\pi i n t/T}.\end{aligned} \tag{3.47}$$

The crucial advantage here is that while $\varphi_T(\boldsymbol{x},t)$ is constrained to be periodic in time, the coefficients $\varphi_n(\boldsymbol{x})$ are completely free. The corresponding differentials are

$$\begin{aligned}D\varphi_T &= \prod_{n=-\infty}^{\infty} D\varphi_n\\ D\rho &= \prod_{n=-\infty}^{\infty} D\rho_n.\end{aligned} \tag{3.48}$$

These substitutions do also include a Jacobian, but this will cancel when we regularize, hence we might as well leave it out right away.

We will now look at how the two factors in (3.46) change when we Fourier expand φ and ρ. First, we consider the integral over ρ. Each term changes as

$$\begin{aligned}\int_{\mathcal{Q}} dt\,\rho(\boldsymbol{x},t)\partial_{\boldsymbol{n}}\varphi(\boldsymbol{x},t) &= \int_Q d\boldsymbol{x}\int_0^T dt\,\sum_m \rho_m e^{2\pi i m t/T}\partial_{\boldsymbol{n}}\sum_n \varphi_n e^{2\pi i n t/T}\\ &= \int_Q d\boldsymbol{x}\,\sum_{m,n}\rho_m\partial_{\boldsymbol{n}}\varphi_n\int_0^T dt\, e^{2\pi i(m+n)t/T}\\ &= T\sum_n\int_Q d\boldsymbol{x}\,\rho_{-n}\partial_{\boldsymbol{n}}\varphi_n,\end{aligned} \tag{3.49}$$

where we have used

$$\int_0^T dt\, e^{2\pi i(m+n)t/T} = T\delta_m^{-n}. \tag{3.50}$$

The action is

$$\begin{aligned} S[\varphi] &= \int_{\mathbb{R}^d} d\boldsymbol{x} \int_0^T dt\, \mathcal{L}[\varphi] = \frac{1}{2}\int_{\mathbb{R}^d} d\boldsymbol{x} \int_0^T dt\, \left(\varphi_t^2 - (\nabla\varphi)^2\right) \\ &= \frac{1}{2}\int_{\mathbb{R}^d} d\boldsymbol{x} \sum_{m,n} \left[\left(\left(\frac{2\pi i m}{T}\varphi_m\right)\left(\frac{2\pi i n}{T}\varphi_n\right) \right.\right. \\ &\qquad\qquad \left.\left. - \nabla\varphi_m \cdot \nabla\varphi_n \right) \int_0^T dt\, e^{2\pi i(m+n)t/T} \right] \\ &= T\sum_n \frac{1}{2}\int_{\mathbb{R}^d} d\boldsymbol{x} \left[\left(\frac{2\pi n}{T}\right)^2 \varphi_n\varphi_{-n} - \nabla\varphi_n \cdot \nabla\varphi_{-n} \right]. \end{aligned} \tag{3.51}$$

Making these substitutions in (3.46), we get

$$\begin{aligned} Z_Q(S,T) &= \int \left(\prod_{n=-\infty}^{\infty} D\varphi_n D\rho_n \right) \exp\left(iS[\varphi] + i\int_Q d\boldsymbol{x}\, \rho\partial_{\boldsymbol{n}}\varphi \right) \\ &= \prod_{n=-\infty}^{\infty} \int D\varphi_n D\rho_n\, e^{iT\bar{S}_n[\varphi_n,\rho_n]}, \end{aligned} \tag{3.52}$$

where

$$\bar{S}_n[\varphi_n,\rho_n] = \frac{1}{2}\int_{\mathbb{R}^d} d\boldsymbol{x}\, \left(\left(\frac{2\pi n}{T}\right)^2 \varphi_n\varphi_{-n} - \nabla\varphi_n\cdot\nabla\varphi_{-n} \right) + \int_Q d\boldsymbol{x}\, \rho_{-n}\partial_{\boldsymbol{n}}\varphi_n. \tag{3.53}$$

Taking the logarithm,

$$\log Z_Q(S,T) = \sum_{n=-\infty}^{\infty} \int D\varphi_n D\rho_n\, e^{iT\bar{S}_n}. \tag{3.54}$$

We will now let $T \to \infty$ in order to treat non-periodic functions, and we wish to change the sum in (3.54) into an integral. Let

$$k_n = \frac{2\pi n}{T}, \tag{3.55}$$

and let $\Delta k = 2\pi/T$. We can write

$$\log Z = \frac{T}{2\pi} \sum_n \Delta k \log \int D\varphi_{k_n} D\rho_{\,k_n} e^{iT\bar{S}_{k_n}}. \tag{3.56}$$

As $T \to \infty$, k_n becomes a continuous variable k and $\Delta k \to 0$, and this becomes a Riemann sum. That is,

$$\log Z \to \frac{T}{2\pi} \int_{-\infty}^{\infty} dk \log \int D\varphi(k) D\rho\,(k) e^{iT\bar{S}(k)}. \tag{3.57}$$

We henceforth drop the argument (k) for the differentials. Split the integral over k into positive and negative parts. Since φ and ρ are real, the eigenfunctions come in complex conjugate pairs, e.g. $\varphi_{-k} = \varphi_k^*$. Therefore,

$$\begin{aligned} \log Z &= \frac{T}{2\pi} \int_0^{\infty} dk \log \int D\varphi D\rho\, e^{iT\bar{S}(k)} \\ &\quad + \frac{T}{2\pi} \int_0^{\infty} dk \log \int D\varphi^* D\rho^* e^{iT\bar{S}(-k)} \\ &= \frac{T}{2\pi} \int_0^{\infty} dk \log \Pi_Q(k), \end{aligned} \tag{3.58}$$

where

$$\Pi_Q(k) = \int D\varphi D\varphi^* D\rho D\rho^*\, e^{iT\hat{S}(k)}, \tag{3.59}$$

and the effective action $\hat{S}$ is

$$\begin{aligned} \hat{S}(k) &= \bar{S}(k) + \bar{S}(-k) \\ &= \int_{\mathbb{R}^d} d\boldsymbol{x} \left(k^2|\varphi|^2 - |\nabla\varphi|^2\right) + \int_Q dA \left(\rho^* \partial_{\boldsymbol{n}} \varphi + \rho \partial_{\boldsymbol{n}} \varphi^*\right). \end{aligned} \tag{3.60}$$

Similarly, we find that

$$\log Z_\infty = \frac{T}{2\pi} \int_0^{\infty} dk \log \Pi_\infty(k), \tag{3.61}$$

where the subscript ∞ indicates that the separation distance between boundaries is infinite.

In order to find the Casimir energy, let $T = -is$. We also perform a Wick rotation, $k = i\kappa$. Inserting this in (3.40) yields

$$\mathcal{E} = -\lim_{s\to\infty} \frac{1}{s} \log \frac{Z_Q(S, -is)}{Z_\infty(-is)} = -\lim_{s\to\infty} \frac{1}{2\pi} \int_0^\infty d\kappa \log \frac{\Pi_Q(i\kappa)}{\Pi_\infty(i\kappa)} \tag{3.62}$$

3.4 Classical Equations of Motion

We now have Π_Q in terms of a functional integral over φ and ρ. In this section, we will resolve the integral over φ. The idea is to decouple φ and ρ so that Π_Q can be written as a functional integral over φ multiplied by one over ρ. The integral over φ will be the same for both Π_Q and Π_∞, and will thus cancel when we divide those.

First we shift φ as

$$\varphi = \phi + \tilde{\varphi}, \tag{3.63}$$

where ϕ is the classical solution as found through the principle of least action, and $\tilde{\varphi}$ is a fluctuating part. The classical action is

$$\hat{S}_{cl} = \int_{\mathbb{R}^d} dV \left(k^2 \phi\phi^* - \nabla\phi \cdot \nabla\phi^*\right) + \int_Q dA \left(\rho^* \partial_{\boldsymbol{n}} \phi + \rho \partial_{\boldsymbol{n}} \phi^*\right). \tag{3.64}$$

Consider a small variation $\delta\phi^*$ in the field. The variation in the action is

$$\delta\hat{S}_{cl} = \int_{\mathbb{R}^d} dV \left(k^2 \phi\delta\phi^* - \nabla\phi \cdot \nabla\delta\phi^*\right) + \int_Q dA \left(\rho \partial_{\boldsymbol{n}} \delta\phi^*\right). \tag{3.65}$$

According to Green's first identity,

$$-\int_V dV\, \nabla\phi \cdot \nabla\delta\phi^* = \int_V dV\, \delta\phi^* \nabla^2\phi - \int_{\partial V} dA\, \delta\phi^* \partial_{\boldsymbol{n}} \phi. \tag{3.66}$$

Adding the cases when V is the interior or exterior of the objects and using the fact that $\int_{\partial V_0} dA = -\int_Q dA$, we get

$$-\int_{\mathbb{R}^d} dV\, \nabla\phi \cdot \nabla\delta\phi^* = \int_{\mathbb{R}^d} dV\, \delta\phi^* \nabla^2\phi - \int_Q dA \left(\delta\phi^*_- \partial_{\boldsymbol{n}} \phi_- - \delta\phi^*_+ \partial_{\boldsymbol{n}} \phi_+\right), \tag{3.67}$$

where the notation means

$$\phi_{\pm}(\boldsymbol{x}) = \lim_{\varepsilon \to 0} \phi(\boldsymbol{x} \pm \varepsilon \boldsymbol{n}), \qquad \boldsymbol{x} \in Q. \tag{3.68}$$

This gives the variation in the classical action

$$\begin{aligned} \delta \hat{S}_{cl} = & \int_{\mathbb{R}^d} dV \left(k^2 \phi + \nabla^2 \phi \right) \delta \phi^* \\ & + \int_Q dA \left(\rho \, \partial_{\boldsymbol{n}} \delta \phi^* - (\delta \phi^*_- \partial_{\boldsymbol{n}} \phi_- - \delta \phi^*_+ \partial_{\boldsymbol{n}} \phi_+) \right), \end{aligned} \tag{3.69}$$

In order for the first integral to vanish, we must have

$$k^2 \phi + \nabla^2 \phi = 0, \tag{3.70}$$

since $\delta\phi$ is arbitrary. With this relation, the variation becomes

$$\delta \hat{S}_{cl} = \int_Q dA \left(\rho \partial_{\boldsymbol{n}} \delta \phi^* - (\delta \phi^*_- \partial_{\boldsymbol{n}} \phi_- - \delta \phi^*_+ \partial_{\boldsymbol{n}} \phi_+) \right). \tag{3.71}$$

The next step is to derive conditions on $\Delta\phi$ and $\Delta\partial_{\boldsymbol{n}}\phi$, where Δ means the jump in the vector field as we move from the outside to inside an object,

$$\Delta \phi = \phi_- - \phi_+. \tag{3.72}$$

Selecting $\delta\phi$ to be continuous across the boundary, we write this as

$$\delta \hat{S}_{cl}[\delta \phi^*] = \int_Q dA \left(\rho \partial_{\boldsymbol{n}} \delta \phi^* - \delta \phi^* \Delta \partial_{\boldsymbol{n}} \phi \right). \tag{3.73}$$

This integral must also be zero for all variations $\delta\phi$.

This is where we run into problems. There are in fact configurations where requiring the integral to be zero imposes conditions on ρ. As an example, consider a configuration where Q_α is a circle with radius R centred at the origin,[†] and select the variation in polar coordinates to be

$$\delta \phi(r, \theta) = \frac{R}{2\pi} \sin\left(\frac{2\pi r}{R} \right) \delta \vartheta(\theta), \tag{3.74}$$

[†]While this is a very special case of configuration consisting on a circle, it serves as a counter-example illustrating that there is a problem here, and it is no stretch to claim the result can be generalized.

where $\delta\vartheta$ is an arbitrary variation that depends only on θ. Note that on Q_α, we have $\delta\phi(R,\theta) = 0$ and

$$\partial_{\boldsymbol{n}}\delta\phi(R,\theta) = \delta\vartheta(\theta), \tag{3.75}$$

so $\delta\hat{S}_{cl}$ becomes

$$\delta\hat{S}_{cl}[\delta\phi^*] = \int_Q dA\,\rho(\theta)\,\delta\vartheta^*(\theta). \tag{3.76}$$

But since $\delta\vartheta(\theta)$ is arbitrary it means that, in order for $\delta\hat{S}_{cl}$ to be zero for all variations $\delta\phi^*$, we must actually have $\rho(\theta) = 0$ everywhere! This issue does not arise for the Dirichlet case, where the integral corresponding to (3.73) is

$$\delta\hat{S}_{\text{Dirichlet}} = \int_Q dA\,(\rho\,\delta\phi^* - \Delta\partial_{\boldsymbol{n}}\phi\,\delta\phi^*)\,, \tag{3.77}$$

which is zero for any ρ and all $\delta\phi^*$ if $\Delta\partial_{\boldsymbol{n}}\phi = \rho$. In [10], it is claimed that (3.60) is indeed the correct expression for the effective action, and the equations of motion for the classical action is derived by analogy with the Dirichlet case and through comparing the two cases to electrostatics. What we have found in this section suggests that if we want to keep ρ unconstrained, there is no classical action that satisfies $\delta\hat{S}_{cl}/\delta\phi^* = 0$, and hence, that the analogy with the Dirichlet case is not sound.

One way to proceed is to disallow variations that place conditions on ρ.[†] If we select the variation to be continuous and such that $\partial_{\boldsymbol{n}}\delta\phi^* = 0$ on Q, (3.71) becomes

$$\delta\hat{S}[\delta\phi^*] = \int_Q dA\,(\partial_{\boldsymbol{n}}\phi_- - \partial_{\boldsymbol{n}}\phi_+)\delta\phi^* = 0, \tag{3.78}$$

which implies that

$$\partial_{\boldsymbol{n}}\phi_- - \partial_{\boldsymbol{n}}\phi_+ = \Delta\partial_{\boldsymbol{n}}\phi = 0, \tag{3.79}$$

which gives a condition on $\Delta\partial_{\boldsymbol{n}}\phi$. Before we continue, we need to discuss how to interpret derivatives of functions at points where they have jump discontinuities.

Suppose the function $f(x)$ has a jump discontinuity of size $\Delta f(a)$ at $x = a$, and that it is continuous for $x \neq a$. We can then write

$$f'(x) = \delta(x-a)\Delta f(a) + f_r'(x), \tag{3.80}$$

[†]It is not really clear what variations we allow and which ones we do not, but the following will require heuristic arguments anyway, so we accept some leniency.

where δ is the Dirac delta function, and f_r is the regular part of f that has the jump singularity removed, i.e.

$$f_r(x) = f(x) - \Delta f(a)\,\theta(x-a), \tag{3.81}$$

where $\theta(x)$ is the step function. This is illustrated in Figure 3.1. Note that if $f_-(a) = f_+(a)$, we have $\Delta f(a) = 0$, and $f(x) = f_r(x)$ everywhere.

With this, (3.79) implies that $\partial_{\boldsymbol{n}}\phi_- = \partial_{\boldsymbol{n}}\phi_+ = \partial_{\boldsymbol{n}}\phi_r$, and we can write (3.71) as

$$\delta\hat{S} = \int_Q dA\,(\rho\partial_{\boldsymbol{n}}\delta\phi^* - \Delta\delta\phi^*\partial_{\boldsymbol{n}}\phi_r)\,. \tag{3.82}$$

We want to apply (3.80) to $\partial_{\boldsymbol{n}}\delta\phi^*$. The jump as we move across the border in the direction of $\boldsymbol{n}$ is $-\Delta\delta\phi^*$, due to the way we defined Δ in (3.72). In order to avoid the actual delta function, suppose the function increases rapidly by an amount $\Delta\delta\phi^*$ in the direction of the normal over a small interval I of radius $1/2\delta_0$, where δ_0 is very large. Then similarly to (3.80), we have

$$\partial_{\boldsymbol{n}}\delta\phi^* = -\delta_0\Delta\delta\phi^* + \partial_{\boldsymbol{n}}\delta\phi_r^*, \tag{3.83}$$

where $\partial_{\boldsymbol{n}}\delta\phi_r^*$ is the derivative just outside the interval I. As $\delta_0 \to \infty$, the interval I becomes a single point and the gradient essentially becomes the Dirac delta function. Selecting $\delta\phi^*$ such that $\partial_{\boldsymbol{n}}\delta\phi_r^* = 0$ (i.e. it is flat except for a jump discontinuity) and inserting this in (3.82), we get

$$\begin{aligned}\delta\hat{S}_{cl} &= \int_Q dA\,(-\rho\delta_0\Delta\delta\phi^* - \Delta\delta\phi^*\partial_{\boldsymbol{n}}\phi_r)\\ &= \int_Q dA\,(-\rho\delta_0 - \partial_{\boldsymbol{n}}\phi_r)\,\Delta\delta\phi^* = 0,\end{aligned} \tag{3.84}$$

which implies

$$\partial_{\boldsymbol{n}}\phi_r = -\delta_0\rho, \tag{3.85}$$

since $\Delta\delta\phi^*$ is arbitrary. Again, we have run into a problem: as $\delta_0 \to \infty$, it seems as ϕ_r has a jump discontinuity across the border. However, ϕ_r is defined as the regular part of ϕ so it cannot have jump discontinuities! In fact, this seems to again suggest $\rho = 0$.

In order to make further progress, we observe that if we replace ϕ_r by ϕ, (3.85) becomes

$$\partial_{\boldsymbol{n}}\phi = -\delta_0\rho. \tag{3.86}$$

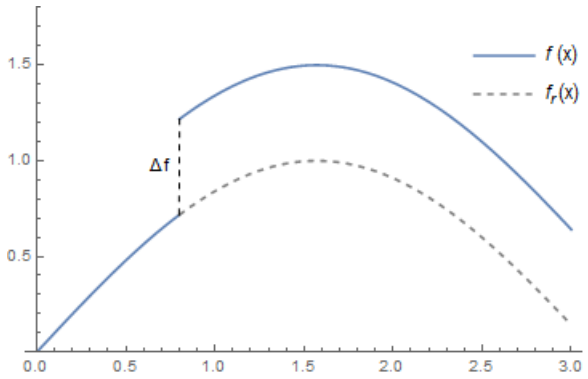


Figure 3.1: The graph of $f(x) = \sin x + 0.5\,\theta(x - 0.8)$. It has a jump discontinuity of size $\Delta f = 0.5$ at $x = 0.8$, so $f'(x) = 0.5\,\delta(x - 0.8) + \cos x$. The regular part $f_r(x) = \sin(x)$ has the jump discontinuity removed.

As this is the same form as (3.83), and it suggests that

$$\Delta \partial_{\boldsymbol{n}} \phi = \rho. \tag{3.87}$$

The argument leading up to this point has been shaky at best, but this last jump does at least appears more reasonable than jumping from the Dirichlet case to the von Neumann case just through analogy. It is possible that applying the full theory of generalized functions will enable us to make this argumentation rigorous. Another way of dealing with this will be presented in Section 3.7.

For now, we accept that the equations of motion are

$$\nabla^2 \phi + k^2 \phi = 0, \quad \boldsymbol{x} \notin Q, \tag{3.88a}$$

$$\Delta \phi = \rho, \qquad \boldsymbol{x} \in Q, \tag{3.88b}$$

$$\Delta \partial_{\boldsymbol{n}} \phi = 0, \qquad \boldsymbol{x} \in Q, \tag{3.88c}$$

and similarly for ϕ^*. This is what was given in [10], except (3.88b) was $\Delta\phi = -\rho$ there. In [10] it was said, in the context of Dirichlet boundary conditions, that "The classical theory defined by $S[\phi, \rho]$, describes a complex scalar field coupled to a set of sources on the surfaces, and is a generalization of electrostatics. By analogy with electrostatics, the field ϕ is continuous throughout space, but its normal derivative jumps by $\rho_\alpha(x)$ across Σ_α." (p. 7). Later, it says that "Neumann boundary conditions are implemented by replacing $\phi(x)$ by $\partial_{\boldsymbol{n}}\phi(x)$ ... Like the Dirichlet case, this case also has an

analogy to electrostatics, namely, a complex field coupled to a set of surface dipole densities $\rho_\alpha(x)$" (p. 19). It seems as [10] is unclear about the nature of ρ in their case. If it is regarded as a dipole of magnitude ρ pointing out of the surface, it is equivalent to a dipole of magnitude $-\rho$ pointing into the surface. It is possible that [10] has interpreted ρ differently than we have, and this is the source of the inconsistency. It is possible that it is just a difference in convention, but that it still describes the same physics.

With these equations of motion, the classical action can be written as a function of only ρ. We need the result that

$$\begin{aligned} -\int_{\mathbb{R}^d} dV\, \nabla\phi\cdot\nabla\phi^* &= \int_{\mathbb{R}^d} dV\, \phi\nabla^2\phi^* - \int_Q dA\left(\phi_-\partial_{\boldsymbol{n}}\phi_-^* - \phi_+\partial_{\boldsymbol{n}}\phi_+^*\right) \\ &= \int_{\mathbb{R}^d} dV\, \phi^*\nabla^2\phi - \int_Q dA\left(\phi_-^*\partial_{\boldsymbol{n}}\phi_- - \phi_+^*\partial_{\boldsymbol{n}}\phi_+\right). \end{aligned} \tag{3.89}$$

Using the equations of motion (3.88), we have $\partial_{\boldsymbol{n}}\phi_- = \partial_{\boldsymbol{n}}\phi_+$ and $\phi_- - \phi_+ = \rho$, and thus

$$\phi_-\partial_{\boldsymbol{n}}\phi_-^* - \phi_+\partial_{\boldsymbol{n}}\phi_+^* = \rho\,\partial_{\boldsymbol{n}}\phi^*. \tag{3.90}$$

With this, we get

$$-\int_{\mathbb{R}^d}\nabla\phi\cdot\nabla\phi^* = \frac{1}{2}\int_{\mathbb{R}^d} dV\left(\phi\nabla^2\phi^* + \phi^*\nabla^2\phi\right) - \frac{1}{2}\int_Q dA\left(\rho\partial_{\boldsymbol{n}}\phi^* + \rho^*\partial_{\boldsymbol{n}}\phi\right). \tag{3.91}$$

Inserting this for the classical action (3.64) gives

$$\begin{aligned} \hat{S}_{cl} &= \frac{1}{2}\int_{\mathbb{R}^d} dV\left(k^2\phi + \nabla^2\phi\right)\phi^* + \frac{1}{2}\int_{\mathbb{R}^d}\left(k^2\phi^* + \nabla^2\phi^*\right)\phi \\ &\quad + \int_Q dA\left(\rho^*\partial_{\boldsymbol{n}}\phi + \rho\partial_{\boldsymbol{n}}\phi^*\right) - \frac{1}{2}\int_Q dA\left(\rho\partial_{\boldsymbol{n}}\phi^* + \rho^*\partial_{\boldsymbol{n}}\phi\right) \\ &= \frac{1}{2}\int_Q dA\left(\rho^*\partial_{\boldsymbol{n}}\phi + \rho\partial_{\boldsymbol{n}}\phi^*\right). \end{aligned} \tag{3.92}$$

Now, back to the effective action. Inserting $\varphi = \phi + \tilde{\varphi}$ in (3.60),

$$\begin{aligned} \hat{S}[\phi+\tilde{\varphi},\rho] = \int_{\mathbb{R}^d} dV\left(k^2(\phi+\tilde{\varphi})(\phi^*+\tilde{\varphi}^*) - \nabla(\phi+\tilde{\varphi})\cdot\nabla(\phi+\tilde{\varphi})\right) \\ + \int_Q dA\left(\rho\partial_{\boldsymbol{n}}(\phi^*+\tilde{\varphi}^*) + \rho^*\partial_{\boldsymbol{n}}(\phi+\tilde{\varphi})\right). \end{aligned} \tag{3.93}$$

The terms that are constant in $\tilde{\varphi}$ correspond to the classical action. The terms linear in $\tilde{\varphi}$ will vanish due to the equations of motion (3.88), and only the quadratic terms will remain,

$$\hat{S} = \hat{S}_{cl} + \int_{\mathbb{R}^d} dV \left(k^2 |\tilde{\varphi}|^2 - |\nabla \tilde{\varphi}|^2 \right). \tag{3.94}$$

Inserting this in (3.59), we get

$$\Pi_Q(k) = \int D\rho D\rho^* e^{s\hat{S}_{cl}} \int D\tilde{\varphi} D\tilde{\varphi}^* e^{s\int_{\mathbb{R}^d} dV \left(|\tilde{\varphi}|^2 - |\nabla \tilde{\varphi}|^2 \right)}. \tag{3.95}$$

The second integral is completely geometry-independent, and will also appear as a factor in $\Pi_\infty(k)$. This means that it will cancel in (3.62). Thus, in practice,

$$\Pi_Q(k) = \int D\rho D\rho^* e^{\frac{1}{2}\int_Q dA(\phi^* \partial_{\boldsymbol{n}} \rho + \phi \partial_{\boldsymbol{n}} \rho^*)}. \tag{3.96}$$

3.5 Scattering Solutions

Let

$$\rho = \sum_{\alpha=1}^{r} \rho_\alpha, \tag{3.97}$$

where ρ_α are functions that are zero everywhere except on Q_α. Then let

$$\varphi_{cl} = \sum_{\alpha} \varphi_\alpha, \tag{3.98}$$

where φ_α is the function satisfying (3.88) when $\rho_\beta = 0$ for all $\beta \neq \alpha$. Also let

$$\hat{S}_{cl} = \frac{1}{2} \sum_{\alpha,\beta} S_{\alpha\beta}, \tag{3.99}$$

where

$$S_{\alpha\beta} = \int_{Q_\alpha} dA \left(\rho_\alpha \partial_{\boldsymbol{n}} \varphi_\beta^* + \rho_\alpha^* \partial_{\boldsymbol{n}} \varphi_\beta \right). \tag{3.100}$$

When the separation distance becomes infinite, only the terms $S_{\alpha\alpha}$ representing the self-interaction terms will remain, and in this case, the classical action becomes

$$\hat{S}_{cl,\infty} = \frac{1}{2} \sum_{\alpha} S_{\alpha\alpha}. \tag{3.101}$$

Let G be the Green's function satisfying

$$(\nabla^2 + k^2)G(\boldsymbol{x};\boldsymbol{\xi}) = \delta(\boldsymbol{x}-\boldsymbol{\xi}). \tag{3.102}$$

The important property of the Green's function is that

$$\varphi_\beta(\boldsymbol{x}_\alpha) = \int_{Q_\beta} d\boldsymbol{\xi}_\beta\, \rho_\beta(\boldsymbol{\xi}_\beta)\partial_{\boldsymbol{n}} G_{\alpha\beta}(\boldsymbol{x}_\alpha;\boldsymbol{\xi}_\beta), \tag{3.103}$$

meaning that $\partial_{\boldsymbol{n}}\varphi_\beta$ can be written as

$$\begin{aligned}\partial_{\boldsymbol{n}}\varphi_\beta(\boldsymbol{x}_\alpha) &= \int_{Q_\beta} d\boldsymbol{\xi}_\beta\, \rho_\beta(\boldsymbol{\xi}_\beta)\partial_{\boldsymbol{nn}} G_{\alpha\beta}(\boldsymbol{x}_\alpha;\boldsymbol{\xi}_\beta)\\ &= \int_{Q_\beta} d\boldsymbol{\xi}_\beta\, \rho_\beta(\boldsymbol{\xi}_\beta) H_{\alpha\beta}(\boldsymbol{x}_\alpha;\boldsymbol{\xi}_\beta),\end{aligned} \tag{3.104}$$

where $H = \partial_{\boldsymbol{nn}} G$.

Introduce a complete set of functions, $\{\psi_\alpha^n\}$ on each of Q_α. We can expand H and ρ_α as

$$H_{\alpha\beta}(\boldsymbol{x}_\alpha;\boldsymbol{\xi}_\beta) = \sum_j H^j_{\alpha\beta}(\boldsymbol{\xi}_\beta)\psi^j_\alpha(\boldsymbol{x}_\alpha) \tag{3.105}$$

and

$$\rho_\alpha(\boldsymbol{x}_\alpha) = \sum_i \rho^i_\alpha \psi^i_\alpha(\boldsymbol{x}_\alpha). \tag{3.106}$$

Using these expansions,

$$\begin{aligned}\partial_{\boldsymbol{n}}\varphi_\beta(\boldsymbol{x}_\alpha) &= \int_{Q_\beta} d\boldsymbol{\xi}_\beta \sum_j \rho^j_\beta \psi^j_\beta(\boldsymbol{\xi}_\beta) \sum_i H^i_{\alpha\beta}(\boldsymbol{\xi}_\beta)\psi^i_\alpha(\boldsymbol{x}_\alpha)\\ &= \sum_{i,j} \rho^j_\beta \psi^i_\alpha(\boldsymbol{x}_\alpha) \int_{Q_\beta} d\boldsymbol{\xi}_\beta\, H^i_{\alpha\beta}(\boldsymbol{\xi}_\beta)\psi^j_\beta(\boldsymbol{\xi}_\beta)\\ &= \sum_{i,j} \psi^i_\alpha(\boldsymbol{x}_\alpha) H^{ij}_{\alpha\beta}\rho^j_\beta,\end{aligned} \tag{3.107}$$

where

$$H^{ij}_{\alpha\beta} = \int_{Q_\beta} d\boldsymbol{\xi}_\beta\, H^i_{\alpha\beta}(\boldsymbol{\xi}_\beta)\psi^j_\beta(\boldsymbol{\xi}_\beta). \tag{3.108}$$

Integrating the second term in (3.100),

$$\begin{aligned}\int_{Q_\alpha} dA\, \rho_\alpha^* \partial_{\boldsymbol{n}} \varphi_\beta &= \int_{Q_\alpha} dA \sum_k \rho_\alpha^{k*} \psi_\alpha^{k*}(\boldsymbol{x}_\alpha) \sum_{i,j} \psi_\alpha^i(\boldsymbol{x}_\alpha) H_{\alpha\beta}^{ij} \rho_\beta^j \\ &= \sum_{i,j,k} H_{\alpha\beta}^{ij} \rho_\beta^j \rho_\alpha^{k*} \int_{Q_\alpha} dA\, \psi_\alpha^{k*}(\boldsymbol{x}_\alpha) \psi_\alpha^i(\boldsymbol{x}_\alpha) \\ &= \sum_{k,j} \rho_\alpha^{k*} \mathcal{H}_{\alpha\beta}^{kj} \rho_\beta^j,\end{aligned} \tag{3.109}$$

where

$$\mathcal{H}_{\alpha\beta}^{kj} = \sum_i H_{\alpha\beta}^{ij} \int_{Q_\alpha} dA\, \psi_\alpha^{k*} \psi_\alpha^i = \sum_i \left\langle \psi_\alpha^k, \psi_\alpha^i \right\rangle H_{\alpha\beta}^{ij}. \tag{3.110}$$

If $\{\psi_\alpha^n\}$ is an orthonormal set of functions, then $\mathcal{H}_{\alpha\beta}^{kj} = H_{\alpha\beta}^{kj}$, but for some configurations it might be convenient to choose a more general function basis. Then

$$S_{\alpha\beta} = \sum_{k,j} \left(\rho_\alpha^{k*} \mathcal{H}_{\alpha\beta}^{kj} \rho_\beta^j + \rho_\alpha^k \mathcal{H}_{\alpha\beta}^{kj*} \rho_\beta^{j*} \right). \tag{3.111}$$

Now the action is

$$\hat{S}_{cl} = \frac{1}{2} \sum_{\alpha,\beta} \sum_{k,j} \rho_\alpha^{k*} \mathcal{H}_{\alpha\beta}^{kj} \rho_\beta^j + \sum_{\alpha,\beta} \sum_{k,j} \rho_\alpha^k \mathcal{H}_{\alpha\beta}^{kj*} \rho_\beta^{j*}. \tag{3.112}$$

For the second term, we can swap the dummy indices k and j as well as α and β to get

$$\hat{S}_l = \frac{1}{2} \sum_{\alpha,\beta} \sum_{k,j} \rho_\alpha^{k*} \left(\mathcal{H}_{\alpha\beta}^{kj} + \mathcal{H}_{\beta\alpha}^{jk*} \right) \rho_\beta^{j*} = \frac{1}{2} \sum_{\alpha,\beta} \sum_{k,j} \rho_\alpha^{k*} U_{\alpha\beta}^{kj} \rho_\beta^j, \tag{3.113}$$

where

$$U_{\alpha\beta}^{kj} = \mathcal{H}_{\alpha\beta}^{kj} + \mathcal{H}_{\beta\alpha}^{jk*}. \tag{3.114}$$

3.6 Discretization and Evaluation

Having $\hat{S}_{cl}$ on the form (3.113), we see that it can actually be written in terms of the inner product,

$$\hat{S}_{cl} = \langle \rho, U\rho \rangle, \tag{3.115}$$

where U is an infinite matrix with entries $U^{ij}_{\alpha\beta}$. We also view $\hat{S}_{cl,\infty}$ as an inner product,

$$\hat{S}_{cl,\infty} = \sum_{\alpha} S_{\alpha\alpha} = \sum_{\alpha} \rho^{k*}_{\alpha} U^{kj}_{\alpha\alpha} \rho^{j}_{\beta} = \langle \rho, U_{\infty} \rho \rangle \,, \tag{3.116}$$

where U_∞ is a block matrix with matrix entries $(U_\infty)_{\alpha\beta} = U_{\alpha\beta}\delta_{\alpha\beta}$, i.e. only the matrices on the diagonal of U. Thus Π_Q becomes

$$\Pi_Q = \int D\rho D\rho^* e^{-\frac{1}{2}s\langle \rho, U\rho\rangle}, \tag{3.117}$$

and similarly for Π_∞.

The matrix U is a block matrix with matrix entries $U_{\alpha\beta}$ for $\alpha, \beta = 1, \ldots, r$, but matrices $U_{\alpha\beta}$ are infinite matrices, as they have one row for each function in the complete set $\{\psi^n_\alpha\}$, which is infinite in general. In order to perform numerical calculations, we must discretize our system and turn these matrices into finite matrices. That is, we will find a finite set of functions $\{\psi^n_\alpha\}_{n=1}^N$ and approximate the matrix elements as a linear combination. We shall now give two examples of how his can be done.

As explained in Section 3.1, one way is to select a Fourier basis. In two dimensions,

$$\psi^n_\alpha(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}, \qquad \theta \in [0, 2\pi). \tag{3.118}$$

This is useful if the configuration is rotationally symmetric. A more general approach is to approximate the object Q_α by a piecewise linear curve with N segments labelled I^n_α. Let L^n_α be the length of that segment. Select the basis

$$\psi^n_\alpha(\boldsymbol{x}) = \begin{cases} 1/\sqrt{L^n_\alpha}, & \boldsymbol{x} \in I^n_\alpha, \\ 0, & \text{otherwise.} \end{cases} \tag{3.119}$$

This basis means we approximate the functions to be constant on each interval. Note that in both these cases, the discretization essentially suppresses high frequencies. In the former case, we explicitly cut off the high frequencies, and in the latter case, the assumption that the functions are constant on each line segment is good unless they contain wavelengths that are smaller than the intervals.

With this choice of basis, we can write Π_Q as

$$\Pi_Q = \mathcal{J} \int \left(\prod_{n=1}^{N} d\rho_n \, d\rho^*_n \right) e^{-\langle \boldsymbol{\rho}, A\boldsymbol{\rho}\rangle}, \tag{3.120}$$

where

$$A = -\frac{1}{2}sU \tag{3.121}$$

and $\mathcal{J}$ is the Jacobian corresponding to the change of variables $D\rho \to d\rho_1 \cdots d\rho_N$. The $\mathcal{J}$ also captures the integration over the trailing variables that were removed through discretization, as described in (3.8). From (3.24), we know that this evaluates to

$$\Pi_Q = \frac{\mathcal{J}}{\det A}. \tag{3.122}$$

Similarly, we have

$$\Pi_\infty = \int D\rho D\rho^* e^{-\langle \rho, A_\infty \rho \rangle} = \frac{\mathcal{J}}{\det A_\infty}, \tag{3.123}$$

where

$$A_\infty = -\frac{1}{2}sU_\infty. \tag{3.124}$$

The Jacobian $\mathcal{J}$ is the same here since we did the performed the same change of variables. Inserting this in (3.62) gives us an expression for the Casimir energy,

$$\mathcal{E} = -\lim_{s\to 0} \frac{1}{2\pi} \int_0^\infty d\kappa \log \frac{\Pi_Q(i\kappa)}{\Pi_\infty(i\kappa)} = \frac{1}{2\pi} \int_0^\infty d\kappa \log \frac{\det A(i\kappa)}{\det A_\infty(i\kappa)}. \tag{3.125}$$

Using the relation

$$\frac{\det A}{\det B} = \det(B^{-1}A), \tag{3.126}$$

we get the final expression for the energy,

$$\mathcal{E} = \frac{1}{2\pi} \int_0^\infty d\kappa \log \det M(i\kappa), \tag{3.127}$$

where $M = A_\infty^{-1} A$.

Summary

Define

$$U^{kj}_{\alpha\beta} = \mathcal{H}^{kj}_{\alpha\beta} + \mathcal{H}^{jk*}_{\beta\alpha} \tag{3.128}$$

for $k, j = 1, \ldots, N$, where

$$\mathcal{H}^{kj}_{\alpha\beta} = \sum_{i=1}^{N} \left\langle \psi^i_\alpha, \psi^k_\alpha \right\rangle \int_{Q_\beta} d\boldsymbol{\xi}_\beta \, H^i_{\alpha\beta}(\boldsymbol{\xi}_\beta)\psi^j_\beta(\boldsymbol{\xi}_\beta), \tag{3.129}$$

and $H^i_{\alpha\beta}$ are the coefficients in the expansion of $H = \partial_{\boldsymbol{nn}} G$,

$$H(\boldsymbol{x}_\alpha; \boldsymbol{\xi}_\beta) = \partial_{\boldsymbol{nn}} G(\boldsymbol{x}_\alpha; \boldsymbol{\xi}_\beta) = \sum_j H^j_{\alpha\beta}(\boldsymbol{\xi}_\beta)\psi^i_\alpha(\boldsymbol{x}_\alpha), \tag{3.130}$$

where G is the Green's function satisfying

$$(\nabla^2 + k^2)G(\boldsymbol{x}; \boldsymbol{\xi}) = \delta(\boldsymbol{x} - \boldsymbol{\xi}), \tag{3.131}$$

evaluated at $k = i\kappa$. Then let the matrix M be a block matrix with matrix elements

$$M_{\alpha\beta} = U^{-1}_{\alpha\alpha} U_{\alpha\beta}, \tag{3.132}$$

and the Casimir energy is given by

$$\mathcal{E} = \frac{1}{2\pi} \int_0^\infty d\kappa \log\det M. \tag{3.133}$$

3.7 FIM with a Gradient Field

In Section 3.4 we found that there are many inconsistencies in deriving the classical equations of motion under von Neumann boundary conditions. The essence of the problem is that in the variation of the classical action,

$$\delta\hat{S}_{cl}[\delta\phi^*] = \int_Q dA \left(\rho\partial_{\boldsymbol{n}}\delta\phi^* - \delta\phi^*\Delta\partial_{\boldsymbol{n}}\phi\right), \tag{3.134}$$

$\delta\phi^*$ and $\partial_{\boldsymbol{n}}\delta\phi^*$ can be made to vanish separately, thus placing conditions on ρ. This does not occur with Dirichlet boundary conditions, as the variation only appears as $\delta\phi^*$.

One approach to this problem is to introduce

$$\boldsymbol{v} = \nabla\varphi, \tag{3.135}$$

giving the boundary condition

$$\boldsymbol{n} \cdot \boldsymbol{v} = 0, \tag{3.136}$$

which now takes the form of a Dirichlet condition. The Lagrangian for $\boldsymbol{v}$ is

$$\mathcal{L}\boldsymbol{v} = \frac{1}{2}\boldsymbol{v}_t^2 - (\nabla\boldsymbol{v})^2. \tag{3.137}$$

The square of the gradient refers to the Frobenius product or the component-wise inner product $\nabla\boldsymbol{v} : \nabla\boldsymbol{v}$, which for two matrices A and B is defined as

$$A : B = \sum_{i,j} A_{ij}B_{ij}. \tag{3.138}$$

The partition function becomes

$$Z_Q(S,T) = \int D\boldsymbol{v}_{C,T,G}\, e^{iS[\varphi]}, \tag{3.139}$$

where the subscript G indicates the condition that $\boldsymbol{v}$ must be a gradient field. This is equivalent to saying it has zero curl everywhere,[†] which we implement this using the delta functional

$$\delta[\nabla\times\boldsymbol{v}|_{\mathbb{R}^d}] = \int D\boldsymbol{\pi}\, e^{i\int_{\mathbb{R}^d} dV \int_0^T dt\, \boldsymbol{\pi}\cdot\nabla\times\boldsymbol{v}}. \tag{3.140}$$

Proceeding as in Section 3.3, the partition function is

$$\log Z = \frac{T}{2\pi}\int_0^\infty dk\, \log\Pi_Q(k) \tag{3.141}$$

[†]In three dimensions, this refers to the usual curl. In two dimensions, we can regard the field as a three-dimensional field with zero z component that is constant in the z-direction, i.e. $\boldsymbol{v}(x,y,z) = (v_x(x,y), v_y(x,y), 0)$. This field has curl $\nabla\times\boldsymbol{v} = (\partial_x v_y - \partial_y v_x)\hat{\boldsymbol{z}}$, giving the condition $\partial_x v_y - \partial_y v_x = 0$. In the following discussion, we assume the dimension is $d = 2$ or $d = 3$.

where

$$\begin{aligned}\Pi_Q(k) &= \int D\boldsymbol{v}\,\delta[\boldsymbol{n}\cdot\boldsymbol{v}|_Q]\,\delta[\nabla\times\boldsymbol{v}|_{\mathbb{R}^d}]\,e^{-iS[\varphi]} \\ &= \int\left(\prod_n D\boldsymbol{v}D\boldsymbol{v}^*D\rho D\rho^*D\boldsymbol{\pi}D\boldsymbol{\pi}^*\right)e^{iT\hat{S}(k)}\end{aligned} \tag{3.142}$$

and the effective action is

$$\begin{aligned}\hat{S} = &\int_{\mathbb{R}^d} dV\left(k^2\boldsymbol{v}\cdot\boldsymbol{v}^* - \nabla\boldsymbol{v}:\nabla\boldsymbol{v}^*\right) \\ &+ \int_Q d\boldsymbol{A}\cdot(\rho^*\boldsymbol{v}+\rho\boldsymbol{v}^*) \\ &+ \int_{\mathbb{R}^d} dV\left(\boldsymbol{\pi}^*\cdot\nabla\times\boldsymbol{v}+\boldsymbol{\pi}\cdot\nabla\times\boldsymbol{v}^*\right).\end{aligned} \tag{3.143}$$

We translate $\boldsymbol{v}$ as $\boldsymbol{v}=\boldsymbol{u}+\tilde{\boldsymbol{v}}$, where $\boldsymbol{u}$ is the classical field satisfying $\delta\hat{S}_{cl}/\delta\boldsymbol{u}=0$. Analogous to (3.67) it can be shown that

$$\int_{\mathbb{R}^d} dV\,\nabla\boldsymbol{u}:\nabla\delta\boldsymbol{u}^* = -\int_{\mathbb{R}^d} dV\,\delta\boldsymbol{u}^*\cdot\nabla^2\boldsymbol{u} + \int_Q dA\,\delta\boldsymbol{u}^*\cdot\Delta\partial_{\boldsymbol{n}}\boldsymbol{u}, \tag{3.144}$$

assuming that $\delta\boldsymbol{u}^*$ has no jump across the boundary. Next, using the property

$$\boldsymbol{a}\cdot\nabla\times\boldsymbol{b} = \boldsymbol{b}\cdot\nabla\times\boldsymbol{a} - \nabla\cdot(\boldsymbol{a}\times\boldsymbol{b}) \tag{3.145}$$

and Gauss' theorem, we have

$$\int_{\mathbb{R}^d} dV\,\boldsymbol{\pi}\cdot\nabla\times\delta\boldsymbol{u}^* = \int_{\mathbb{R}^d} dV\,\delta\boldsymbol{u}^*\cdot\nabla\times\boldsymbol{\pi} - \int_Q dA\,\boldsymbol{n}\cdot(\Delta\boldsymbol{\pi}\times\delta\boldsymbol{u}^*). \tag{3.146}$$

Next, using the property

$$\boldsymbol{n}\cdot(\boldsymbol{a}\times\boldsymbol{b}) = (\boldsymbol{n}\times\boldsymbol{a})\cdot\boldsymbol{b} \tag{3.147}$$

and assuming $\delta\boldsymbol{u}^*$ is continuous across the boundary, the second integral becomes

$$\int_Q dA\,\boldsymbol{n}\cdot(\boldsymbol{\pi}\times\delta\boldsymbol{u}^*) = \int_Q dA\,\delta\boldsymbol{u}^*\cdot(\boldsymbol{n}\times\Delta\boldsymbol{\pi}). \tag{3.148}$$

Then the variation in the classical action is

$$\begin{aligned}\delta\hat{S}_{cl} = &\int_{\mathbb{R}^d} dV \left(k^2\boldsymbol{u} + \nabla^2\boldsymbol{u} + \nabla\times\boldsymbol{\pi}\right)\cdot\delta\boldsymbol{u}^* \\ &+ \int_Q dA\,(\boldsymbol{n}\rho - \Delta\partial_{\boldsymbol{n}}\boldsymbol{u} - \boldsymbol{n}\times\delta\boldsymbol{\pi})\cdot\delta\boldsymbol{u}^*.\end{aligned} \tag{3.149}$$

This yields the equations of motion,

$$\nabla^2\boldsymbol{u} + k^2\boldsymbol{u} = -\nabla\times\boldsymbol{\pi}, \quad \boldsymbol{x}\notin Q, \tag{3.150a}$$

$$\Delta\boldsymbol{u} = 0, \quad \boldsymbol{x}\in Q, \tag{3.150b}$$

$$\Delta\partial_{\boldsymbol{n}}\boldsymbol{u} = \boldsymbol{n}\rho - \boldsymbol{n}\times\Delta\boldsymbol{\pi}, \quad \boldsymbol{x}\in Q. \tag{3.150c}$$

This resolves the issue we had before, since $\partial_{\boldsymbol{n}}\delta\boldsymbol{u}^*$ does not appear in (3.149).

Using the equations of motion, the classical action can be simplified to

$$\hat{S}_{cl} = \frac{1}{2}\int_{\mathbb{R}^d} dV\,\boldsymbol{u}^*\cdot\nabla\times\boldsymbol{\pi} + \frac{1}{2}\int_Q dA\,(\boldsymbol{n}\rho - \boldsymbol{n}\times\Delta\boldsymbol{\pi})\cdot\boldsymbol{u}^* + c.c., \tag{3.151}$$

where *c.c.* indicates the complex conjugate of the preceding terms. The effective action can be written as

$$\hat{S} = \hat{S}_{cl} + \int_{\mathbb{R}^d} dV\left(|\tilde{\boldsymbol{v}}|^2 - |\nabla\tilde{\boldsymbol{v}}|^2\right). \tag{3.152}$$

The second integral is geometry-independent and will vanish when we regularize. It is also possible to extract the parts containing $\boldsymbol{\pi}$, leaving only $\Delta\boldsymbol{\pi}$. If we let $\boldsymbol{j} = \Delta\boldsymbol{\pi}$, we can write

$$\begin{aligned}\Pi_Q(k) = &\int D\boldsymbol{\pi}D\boldsymbol{\pi}\, e^{\frac{1}{2}s\int_{\mathbb{R}^d} dV\,(\boldsymbol{u}^*\cdot\nabla\times\boldsymbol{\pi} + \boldsymbol{u}\cdot\nabla\times\boldsymbol{\pi}^*)} \\ &\int D\rho D\rho^* D\boldsymbol{j}D\boldsymbol{j}^*\, e^{\frac{1}{2}s\int_Q dA\,(\boldsymbol{n}\rho - \boldsymbol{n}\times\boldsymbol{j})\cdot\boldsymbol{u}^* + c.c.}.\end{aligned} \tag{3.153}$$

Again, the first integral is geometry-independent and will be found as a factor in Π_∞ as well, meaning it will cancel. In practice, we end up with

$$\Pi_Q(k) = \int D\rho D\rho^* D\boldsymbol{j}D\boldsymbol{j}^*\, e^{s\hat{S}_{\text{eff}}} \tag{3.154}$$

where

$$\hat{S}_{\text{eff}} = \frac{1}{2}\int_Q dA\,(\boldsymbol{n}\rho - \boldsymbol{n}\times\boldsymbol{j})\cdot\boldsymbol{u}^* + c.c. \tag{3.155}$$

We have solved the issue we had before, but at a cost. Since we now have a vector equation, the Green's function G becomes a matrix. Furthermore, the fact that we now have the inhomogeneous Helmholtz equation means the integral identities we derive to express $\boldsymbol{v}$ in terms of G will contain integrals over all V_α. Consequently, we must evaluate a d-dimensional integral, unlike for the Dirichlet case where we had a $(d-1)$-dimensional integral. This causes a massive increase in computation cost.

Chapter 4

The boundary integral method

In this chapter we will develop the boundary integral method with von Neumann boundary conditions. In Section 4.1 we will show that the pressure at a point $\boldsymbol{x}$ on the boundary Q_α can be written as

$$p(\boldsymbol{x}) = \lim_{\boldsymbol{x}' \to \boldsymbol{x}} \int_0^\infty \frac{d\omega}{2\pi} \left(\partial_{\boldsymbol{t}\boldsymbol{t}'} + \omega^2\right) D(\boldsymbol{x}, \boldsymbol{x}', \omega), \tag{4.1}$$

where D is the Green's function satisfying

$$(\nabla^2 + \omega^2) D(\boldsymbol{x}, \boldsymbol{x}', \omega) = \delta(\boldsymbol{x} - \boldsymbol{x}), \tag{4.2}$$

with boundary conditions

$$\begin{aligned} \boldsymbol{n} \cdot \nabla_{\boldsymbol{x}} D(\boldsymbol{x}, \boldsymbol{x}', \omega) = 0, \qquad & \boldsymbol{x} \in Q_\alpha, \\ \boldsymbol{n}' \cdot \nabla_{\boldsymbol{x}'} D(\boldsymbol{x}, \boldsymbol{x}', \omega) = 0, \qquad & \boldsymbol{x}' \in Q_\alpha. \end{aligned} \tag{4.3}$$

This integral applies for general dimensions, but for the rest of the section, we will limit the discussion to two-dimensional space. In Section 4.2 find an equation for D in terms of a boundary integral problem. In Section 4.3 we discretize these equations, turn them into a system of linear equations. During this work we introduce the notion of self-interaction, which is discussed more closely in Section 4.4. In Section 4.5 and Section 4.6 we show how this is applied to parallel plates and concentric circles respectively. Then we will in Section 4.7 look at how the computations can be simplified through symmetries. Finally, in Section 4.8 we study the simplified case of one-dimensional space.

For someone unacquainted with the boundary integral method, it might be wise to start with Section 4.8, which presents a simplified version of BIM. Next I recommend starting from Section 4.2 and skipping Section 4.1 as this section contains rather complicated calculations that are not important for the rest of the discussion. The important result from Section 4.1 is the integral (4.1), and most of the discussion will be about finding the Green's function D.

4.1 The force integral

From the introduction, the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\varphi_\mu\varphi_\nu. \tag{4.4}$$

In developing the boundary integral method, we will also need the stress energy tensor,

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial^\nu\varphi - \delta^\mu_\nu\mathcal{L}. \tag{4.5}$$

Noether's theorem says that this is conserved,

$$\partial_\mu T^{\mu\nu} = 0. \tag{4.6}$$

This gives us a further set of equations. The case $\nu = 0$ corresponds to conservation of energy, and the cases $\nu = 1, 2, 3$ each correspond to conservation of one component of momentum. Explicitly, the conservation of momentum can be written as

$$\partial_t\boldsymbol{p} + \nabla\cdot S = 0, \tag{4.7}$$

where

$$\boldsymbol{p} = \varphi_t\nabla\varphi \tag{4.8}$$

is the momentum density and

$$S = -\nabla\varphi\nabla\varphi + \frac{1}{2}Tr(\nabla\varphi\nabla\varphi)I - \frac{1}{2}\varphi_t^2 I \tag{4.9}$$

is the momentum flux.

Force is defined as the time derivative of momentum. Using this fact, the equation of momentum conservation (4.7) and the divergence theorem, we can write the force on object α as

$$\boldsymbol{F}_\alpha = \partial_t \int_{V_\alpha} dV\, \boldsymbol{p}(\boldsymbol{x},t) = -\int_{V_\alpha} dV\, \nabla\cdot S(\boldsymbol{x},t) = -\oint_{Q_\alpha} dA\, \boldsymbol{n}\cdot S(\boldsymbol{x},t), \quad (4.10)$$

where $\boldsymbol{n}$ is the normal on the surface Q_α.

We will now quantize he scalar field $\hat{\varphi}$. The quantized field follows the canonical commutation relations

$$\begin{aligned} &[\hat{\varphi}(\boldsymbol{x},t), \hat{\varphi}(\boldsymbol{x}',t)] = 0, \\ &[\hat{\varphi}_t(\boldsymbol{x},t), \hat{\varphi}_t(\boldsymbol{x}',t)] = 0, \\ &[\hat{\varphi}_t(\boldsymbol{x},t), \hat{\varphi}(\boldsymbol{x}',t)] = i\delta(\boldsymbol{x}-\boldsymbol{x}'). \end{aligned} \quad (4.11)$$

Now, we presciently perform a Wick rotation, transforming to imaginary time $s = it$. The time derivative changes as $\partial_t = \frac{ds}{dt}\partial_s = i\partial_s$. The canonical commutation relations become

$$\begin{aligned} &[\hat{\varphi}(\boldsymbol{x},s), \hat{\varphi}(\boldsymbol{x}',s)] = 0, \\ &[\hat{\varphi}_s(\boldsymbol{x},s), \hat{\varphi}_s(\boldsymbol{x}',s)] = 0, \\ &[\hat{\varphi}_s(\boldsymbol{x},s), \hat{\varphi}(\boldsymbol{x}',s)] = \delta(\boldsymbol{x}-\boldsymbol{x}'). \end{aligned} \quad (4.12)$$

Observe that the stress tensor can be written as

$$\begin{aligned} S(\boldsymbol{x},s) &= \left(-\nabla\nabla + \frac{1}{2}\operatorname{Tr}(\nabla\nabla)I + \frac{1}{2}\partial_s^2 I\right)\varphi(x,s)\varphi(x,s) \\ &= \left(-\nabla\nabla + \frac{1}{2}\operatorname{Tr}(\nabla\nabla)I + \frac{1}{2}\partial_s^2 I\right)\frac{1}{2}\{\varphi(\boldsymbol{x},s), \varphi(\boldsymbol{x},s)\} \end{aligned} \quad (4.13)$$

where $\{A,B\} = AB + BA$ is the anti-commutator.

We define the quantum stress tensor S_q by the point separation method [6]. This method evaluates one of the fields at some nearby point $(\boldsymbol{x}', s')$ and lets it approach $(\boldsymbol{x}, s)$. The corresponding time and gradient operators are replaced by $\partial_{s'}$ and ∇' respectively. Then

$$\begin{aligned} S_q(\boldsymbol{x},s) &= \lim_{\substack{\boldsymbol{x}'\to\boldsymbol{x} \\ s'\to s}} \left(-\nabla\nabla' + \frac{1}{2}\operatorname{Tr}(\nabla\nabla')I + \frac{1}{2}\partial_s\partial_{s'}I\right)\frac{1}{2}\left\langle\{\varphi(\boldsymbol{x},s), \varphi(\boldsymbol{x}',s')\}\right\rangle \\ &= \lim_{\substack{\boldsymbol{x}'\to\boldsymbol{x} \\ s'\to s}} \left(-\nabla\nabla' + \frac{1}{2}\operatorname{Tr}(\nabla\nabla')I + \frac{1}{2}\partial_s\partial_{s'}I\right)\frac{1}{2}D^{(1)}(\boldsymbol{x},s,\boldsymbol{x}',s'), \end{aligned} \quad (4.14)$$

where $D^{(1)}(\boldsymbol{x}, s, \boldsymbol{x}', s)$ is the Hadamard's Green's function.

Consider the Heaviside step function

$$\theta(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & x = 0 \\ 1, & x > 0 \end{cases} \tag{4.15}$$

This has the property $\theta(x) + \theta(-x) = 1$. Using this property, we can write $D^{(1)}$ as

$$\begin{aligned} D^{(1)}(\boldsymbol{x}, s, \boldsymbol{x}', s) &= (\theta(s - s') + \theta(s' - s)) \langle \{\hat{\varphi}(\boldsymbol{x}, s), \hat{\varphi}(\boldsymbol{x}', s')\} \rangle \\ &= 2 \left(\theta(s - s') \langle \hat{\varphi}(\boldsymbol{x}, s) \hat{\varphi}(\boldsymbol{x}', s') \rangle + \theta(s' - s) \langle \hat{\varphi}(\boldsymbol{x}', s') \hat{\varphi}(\boldsymbol{x}, s) \rangle \right) \\ &\quad - \theta(s - s') \langle [\hat{\varphi}(\boldsymbol{x}, s), \hat{\varphi}(\boldsymbol{x}', s')] \rangle \\ &\quad + \theta(s' - s) \langle [\hat{\varphi}(\boldsymbol{x}, s), \hat{\varphi}(\boldsymbol{x}', s')] \rangle \\ &= 2D(\boldsymbol{x}, s, \boldsymbol{x}', s') - D^R(\boldsymbol{x}, s, \boldsymbol{x}', s') - D^A(\boldsymbol{x}, s, \boldsymbol{x}', s'), \end{aligned} \tag{4.16}$$

where

$$D^R(\boldsymbol{x}, s, \boldsymbol{x}', s') = \theta(s - s') \langle [\hat{\varphi}(\boldsymbol{x}, s), \hat{\varphi}(\boldsymbol{x}', s')] \rangle \tag{4.17}$$

$$D^A(\boldsymbol{x}, s, \boldsymbol{x}', s') = -\theta(s' - s) \langle [\hat{\varphi}(\boldsymbol{x}, s), \hat{\varphi}(\boldsymbol{x}', s')] \rangle \tag{4.18}$$

are the retarded and advanced Green's functions respectively, and

$$\begin{aligned} D(\boldsymbol{x}, s, \boldsymbol{x}', s') &= \theta(s - s') \langle \hat{\varphi}(\boldsymbol{x}, s) \hat{\varphi}(\boldsymbol{x}', s') \rangle + \theta(s' - s) \langle \hat{\varphi}(\boldsymbol{x}', s') \hat{\varphi}(\boldsymbol{x}, s) \rangle \\ &= \langle \mathcal{T}(\hat{\varphi}(\boldsymbol{x}, s) \hat{\varphi}(\boldsymbol{x}', s')) \rangle , \end{aligned} \tag{4.19}$$

where $\mathcal{T}$ represents time-ordering. Because of the commutation relations (4.12), D^R and D^A vanish in the limit $(\boldsymbol{x}', s') \to (\boldsymbol{x}, s)$. Thus,

$$S_q(\boldsymbol{x}, s) = \lim_{\substack{\boldsymbol{x}' \to \boldsymbol{x} \\ s' \to s}} \left(-\nabla \nabla' + \frac{1}{2} \operatorname{Tr}(\nabla \nabla') I + \frac{1}{2} \partial_s \partial_{s'} \right) D(\boldsymbol{x}, s, \boldsymbol{x}', s'). \tag{4.20}$$

We will now derive some properties for D and show that it is a Green's function. This is known as the basic Green's function, and we will refer to this as *the* Green's function. Assume that the quantum field is in thermal

equilibrium. For an operator $\hat{A}$ in this state, it is known that $\langle \hat{A} \rangle = \mathrm{Tr}(\hat{\rho}\hat{A})$, where the density matrix $\hat{\rho}$ is

$$\hat{\rho} = \frac{1}{Z} e^{-\beta \hat{H}}, \tag{4.21}$$

where $\beta = 1/T$, T is temperature, $Z = \mathrm{Tr}\left(e^{-\beta \hat{H}}\right)$ is the partition function and $\hat{H}$ is the Hamiltonian. Then the Green's function can be written as

$$D(\boldsymbol{x}, s, \boldsymbol{x}', s') = \frac{1}{Z} \mathrm{Tr}\left(e^{-\beta \hat{H}} \mathcal{T}(\hat{\varphi}(\boldsymbol{x}, s)\hat{\varphi}(\boldsymbol{x}', s'))\right). \tag{4.22}$$

In the following, we study properties pertaining to the time domain so for simplicity we abbreviate

$$D(s, s') = D(\boldsymbol{x}, s, \boldsymbol{x}', s'). \tag{4.23}$$

Let

$$\begin{aligned} D^+(s, s') &= \langle \hat{\varphi}(\boldsymbol{x}, s)\hat{\varphi}(\boldsymbol{x}', s') \rangle \\ D^-(s, s') &= \langle \hat{\varphi}(\boldsymbol{x}', s')\hat{\varphi}(\boldsymbol{x}, s) \rangle, \end{aligned} \tag{4.24}$$

and note that the Green's function can be written as

$$D(s, s') = \begin{cases} D^+(s, s'), & s > s' \\ D^-(s, s'), & s < s' \end{cases}. \tag{4.25}$$

We will first show that the Green's function is dependent only of $s - s'$. In the Heisenberg picture, $\hat{\varphi}$ satisfies the equation of motion

$$\frac{d}{ds}\hat{\varphi}(\boldsymbol{x}, s) = [\hat{H}, \hat{\varphi}]. \tag{4.26}$$

Assuming the conductors are stationary so that $\hat{H}$ is stationary, this has the solution

$$\hat{\varphi}(\boldsymbol{x}, s) = e^{s\hat{H}} \hat{\varphi}(\boldsymbol{x}) e^{-s\hat{H}}, \tag{4.27}$$

where $\hat{\varphi}(\boldsymbol{x}) = \hat{\varphi}(\boldsymbol{x}, 0)$. Using this property, we observe that

$$\begin{aligned} D^+(s + \beta, s') &= \frac{1}{Z} \mathrm{Tr}\left(e^{-\beta \hat{H}} e^{(s+\beta)\hat{H}} \hat{\varphi}(\boldsymbol{x}) e^{-(s+\beta)\hat{H}} \hat{\varphi}(\boldsymbol{x}', s')\right) \\ &= \frac{1}{Z} \mathrm{Tr}\left(e^{s\hat{H}} \hat{\varphi}(\boldsymbol{x}) e^{-s\hat{H}} e^{-\beta \hat{H}} \hat{\varphi}(\boldsymbol{x}', s')\right) \\ &= \frac{1}{Z} \mathrm{Tr}\left(\hat{\varphi}(\boldsymbol{x}, s) e^{-\beta \hat{H}} \hat{\varphi}(\boldsymbol{x}', s')\right). \end{aligned} \tag{4.28}$$

Using the property of matrices

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB), \tag{4.29}$$

we get

$$D^+(s+\beta, s') = \frac{1}{Z}\,\mathrm{Tr}\left(e^{-\beta\hat{H}}\hat{\varphi}(\boldsymbol{x}', s')\hat{\varphi}(\boldsymbol{x}, s)\right) = D^-(s, s'). \tag{4.30}$$

Furthermore,

$$\begin{aligned} D^+(s, s') &= \frac{1}{Z}\,\mathrm{Tr}\left(e^{-\beta\hat{H}}\hat{\varphi}(\boldsymbol{x}, s)\hat{\varphi}(\boldsymbol{x}', s')\right) \\ &= \frac{1}{Z}\,\mathrm{Tr}\left(e^{-\beta\hat{H}}e^{s\hat{H}}\hat{\varphi}(\boldsymbol{x})e^{-s\hat{H}}e^{s'\hat{H}}\hat{\varphi}(\boldsymbol{x}')e^{-s'\hat{H}}\right). \end{aligned} \tag{4.31}$$

Again using (4.29),

$$\begin{aligned} D^+(s, s') &= \frac{1}{Z}\,\mathrm{Tr}\left(e^{-s'\hat{H}}e^{-\beta\hat{H}}e^{s\hat{H}}\hat{\varphi}(\boldsymbol{x})e^{-s\hat{H}}e^{s'\hat{H}}\hat{\varphi}(\boldsymbol{x}')\right) \\ &= \frac{1}{Z}\,\mathrm{Tr}\left(e^{-\beta\hat{H}}e^{(s-s')\hat{H}}\hat{\varphi}(\boldsymbol{x})e^{-(s-s')\hat{H}}\hat{\varphi}(\boldsymbol{x}')\right) \\ &= \frac{1}{Z}\,\mathrm{Tr}\left(e^{-\beta\hat{H}}\hat{\varphi}(\boldsymbol{x}, s-s')\hat{\varphi}(\boldsymbol{x}', 0)\right) \\ &= D^+(s-s', 0). \end{aligned} \tag{4.32}$$

Similarly, it can be shown that

$$D^-(s, s'+\beta) = D^+(s, s') \tag{4.33}$$

and

$$D^-(s, s') = D^-(s-s', 0). \tag{4.34}$$

(4.30) and (4.33) are known as the Kubo-Martin-Schwinger (KMS) boundary conditions [16] [18]. Based on these properties, we define

$$D(s) = \begin{cases} D^+(s, 0), & s > 0 \\ D^-(s, 0) & s < 0 \end{cases} \tag{4.35}$$

and note that $D(s, s') = D(s - s')$. Also note that with this convention, the stress tensor becomes

$$S_q(\boldsymbol{x}, s) = \lim_{\substack{\boldsymbol{x}' \to \boldsymbol{x} \\ s \to 0}} \left(-\nabla\nabla' + \frac{1}{2}\,\mathrm{Tr}(\nabla\nabla')I - \frac{1}{2}\partial_{ss}\right) D(\boldsymbol{x}, \boldsymbol{x}', s), \tag{4.36}$$

and in particular note that the partial derivative has changed as $\partial_s \partial_{s'} \to -\partial_{ss}$.

Next, let $|n\rangle$ be a complete set of eigenstates for $\hat{H}$,

$$\hat{H}\,|n\rangle = E_n\,|n\rangle\,. \tag{4.37}$$

For $s > 0$ we have

$$\begin{aligned} D^+(s,0) &= \frac{1}{Z}\,\mathrm{Tr}\left(e^{-\beta\hat{H}}e^{s\hat{H}}\hat{\varphi}(\boldsymbol{x})e^{-s\hat{H}}\hat{\varphi}(\boldsymbol{x}')\right) \\ &= \frac{1}{Z}\sum_n \left\langle n\middle|e^{-\beta\hat{H}}e^{s\hat{H}}\hat{\varphi}(\boldsymbol{x})e^{-s\hat{H}}\hat{\varphi}(\boldsymbol{x}')\middle|n\right\rangle \\ &= \frac{1}{Z}\sum_{n,n'} \left\langle n\middle|e^{-\beta E_n}e^{sE_n}\hat{\varphi}(\boldsymbol{x})\middle|n'\right\rangle \left\langle n'\middle|e^{-sE_{n'}}\hat{\varphi}(\boldsymbol{x}')\middle|n\right\rangle \\ &= \frac{1}{Z}\sum_{n,n'} e^{-(\beta-s)E_n}e^{-sE_{n'}}\,\langle n|\hat{\varphi}(\boldsymbol{x})|n'\rangle\,\langle n'|\hat{\varphi}(\boldsymbol{x}')|n\rangle\,. \end{aligned} \tag{4.38}$$

Notice that the terms grow without bound when $s > \beta$, so $D(s)$ exists only for $s \le \beta$. Similarly, for $s < 0$,

$$D^-(s,0) = \frac{1}{Z}\sum_{n,n'} e^{-(\beta+s)E_n}e^{-sE_{n'}}\,\langle n|\hat{\varphi}(\boldsymbol{x})|n'\rangle\,\langle n'|\hat{\varphi}(\boldsymbol{x}')|n\rangle\,, \tag{4.39}$$

which imposes the restriction $s \ge -\beta$. Thus $D(s)$ exists only for $s \in [-\beta, \beta]$. Applying the KMS conditions gives

$$D(s+\beta) = D^+(s+\beta,0) = D^-(s,0) = D(s), \tag{4.40}$$

which shows that $D(s)$ is determined by only its values on $[-\beta, 0]$. We therefore extend the definition of $D(s)$ by letting it be $D^-(s,0)$ on $[-\beta,0]$, and β-periodic elsewhere.

Finally we will show that $D(s)$ is a Green's function for the operator $\partial_{ss} + \nabla^2$. First we write

$$D(s) = \theta(s)\,\langle\hat{\varphi}(\boldsymbol{x},s)\hat{\varphi}(\boldsymbol{x}',0)\rangle + \theta(-s)\,\langle\hat{\varphi}(\boldsymbol{x}',0)\hat{\varphi}(\boldsymbol{x},s)\rangle\,, \tag{4.41}$$

where θ is the Heaviside step function. Using $\theta'(s) = \delta(s)$,

$$\begin{aligned} D'(s) = {}& \delta(s)\,\langle\hat{\varphi}(\boldsymbol{x},s)\hat{\varphi}(\boldsymbol{x}',0)\rangle - \delta(s)\,\langle\hat{\varphi}(\boldsymbol{x}',0)\hat{\varphi}(\boldsymbol{x},s)\rangle \\ &+ \theta(s)\,\langle\partial_s\hat{\varphi}(\boldsymbol{x},s)\hat{\varphi}(\boldsymbol{x}',0)\rangle + \theta(-s)\,\langle\hat{\varphi}(\boldsymbol{x}',0)\partial_s\hat{\varphi}(\boldsymbol{x},s)\rangle\,. \end{aligned} \tag{4.42}$$

Then the commutation relations (4.12) lead to

$$D'(s) = \theta(s)\left\langle \hat{\varphi}_s(\boldsymbol{x}, s)\hat{\varphi}(\boldsymbol{x}', 0)\right\rangle + \theta(-s)\left\langle \hat{\varphi}(\boldsymbol{x}', 0)\hat{\varphi}_s(\boldsymbol{x}, s)\right\rangle . \tag{4.43}$$

A second differentiation gives

$$\begin{aligned} D''(s) = {} & \delta(s)\left\langle \hat{\varphi}_s(\boldsymbol{x}, s)\hat{\varphi}(\boldsymbol{x}', 0)\right\rangle - \delta(s)\left\langle \hat{\varphi}(\boldsymbol{x}', 0)\hat{\varphi}_s(\boldsymbol{x}, s)\right\rangle \\ & + \theta(s)\left\langle \partial_s\hat{\varphi}_s(\boldsymbol{x}, s)\hat{\varphi}(\boldsymbol{x}', 0)\right\rangle + \theta(-s)\left\langle \hat{\varphi}(\boldsymbol{x}', 0)\partial_s\hat{\varphi}_s(\boldsymbol{x}, s)\right\rangle . \end{aligned} \tag{4.44}$$

Here the property $\delta(s)f(s) = \delta(s)f(0)$ lets us use the commutation relation $[\hat{\varphi}_s(\boldsymbol{x}, 0), \hat{\varphi}(\boldsymbol{x}', 0)] = \delta(\boldsymbol{x} - \boldsymbol{x}')$. Furthermore, $\hat{\varphi}(\boldsymbol{x}, s)$ satisfies the Helmholtz equation,

$$\partial_{ss}\varphi(\boldsymbol{x}, s) + \nabla^2\varphi(\boldsymbol{x}, s) = 0. \tag{4.45}$$

Then

$$\partial_{ss}D(\boldsymbol{x}, \boldsymbol{x}', s) + \nabla^2 D(\boldsymbol{x}, \boldsymbol{x}', s) = \delta(s)\delta(\boldsymbol{x} - \boldsymbol{x}'), \tag{4.46}$$

which shows that D is indeed a Green's function for (4.45).

Our boundary conditions on $\hat{\varphi}$ are $\partial_{\boldsymbol{n}}\hat{\varphi}|_Q = 0$. Using this in the definition of D, (4.19), we see that

$$\begin{aligned} \partial_{\boldsymbol{n}}D(\boldsymbol{x}, \boldsymbol{x}', s) = 0, \qquad & \boldsymbol{x} \in Q \\ \partial_{\boldsymbol{n}'}D(\boldsymbol{x}, \boldsymbol{x}', s) = 0, \qquad & \boldsymbol{x}' \in Q, \end{aligned} \tag{4.47}$$

where $\partial_{\boldsymbol{n}}$ and $\partial_{\boldsymbol{n}'}$ are the derivatives in the direction of the normal of Q at points $\boldsymbol{x}$ and $\boldsymbol{x}'$ respectively.

We showed that D is periodic with period $\beta = 1/T$. In this thesis we are concerned with the case $T \to 0$ or $\beta \to \infty$, so $D(\boldsymbol{x}, \boldsymbol{x}', s)$ can be Fourier transformed in s. We obtain the equation

$$\nabla^2 D(\boldsymbol{x}, \boldsymbol{x}', \omega) - \omega^2 D(\boldsymbol{x}, \boldsymbol{x}', \omega) = \delta(\boldsymbol{x} - \boldsymbol{x}') \tag{4.48a}$$

$$\partial_{\boldsymbol{n}}D(\boldsymbol{x}, \boldsymbol{x}', \omega) = 0, \qquad \boldsymbol{x} \in Q \tag{4.48b}$$

$$\partial_{\boldsymbol{n}'}D(\boldsymbol{x}, \boldsymbol{x}', \omega) = 0, \qquad \boldsymbol{x}' \in Q. \tag{4.48c}$$

We have found that the force on object α is

$$\boldsymbol{F}_\alpha = \int_{Q_\alpha} d\boldsymbol{A} \cdot \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_q(\boldsymbol{x}, \omega) \tag{4.49}$$

where S_q is the stress tensor

$$S_q(\boldsymbol{x},\omega) = \lim_{\boldsymbol{x}'\to\boldsymbol{x}} \left(-\nabla\nabla' + \frac{1}{2}\operatorname{Tr}(\nabla\nabla') + \frac{1}{2}\omega^2 \right) D(\boldsymbol{x},\boldsymbol{x}',\omega) \tag{4.50}$$

and D is the Green's function of $L = \nabla^2 - \omega^2$ satisfying the boundary conditions.

In two dimensions, the gradient can be decomposed into normal and tangential components near the surface Q,

$$\nabla = \boldsymbol{n}\,\partial_{\boldsymbol{n}} + \boldsymbol{t}\,\partial_{\boldsymbol{t}}. \tag{4.51}$$

In terms of these partial derivatives, the trace of the gradients is

$$\operatorname{Tr}(\nabla\nabla') = \partial_{\boldsymbol{n}\boldsymbol{n}'} + \partial_{\boldsymbol{t}\boldsymbol{t}'}. \tag{4.52}$$

Because of the boundary condition, terms containing $\partial_{\boldsymbol{n}}$ vanish on the boundary. The stress tensor is then

$$S_q(\boldsymbol{x},\omega) = \lim_{\boldsymbol{x}'\to\boldsymbol{x}} \left(\boldsymbol{t}\boldsymbol{t}'\,\partial_{\boldsymbol{t}\boldsymbol{t}'} + \frac{1}{2}\partial_{\boldsymbol{t}\boldsymbol{t}'} + \frac{1}{2}\omega^2 \right) D(\boldsymbol{x},\boldsymbol{x}',\omega) \tag{4.53}$$

The product $\boldsymbol{n}\cdot\boldsymbol{t}\boldsymbol{t}' = 0$, and so the force can be written as

$$\boldsymbol{F}_\alpha = -\int_{Q_\alpha} dA\,\boldsymbol{n}\cdot\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}(\partial_{\boldsymbol{t}\boldsymbol{t}'} + \omega^2)D(\boldsymbol{x},\boldsymbol{x},\omega), \tag{4.54}$$

where we have dropped the limit and interpret $\partial_{\boldsymbol{t}'}$ as working on the second argument of D. As we shall see, D depends only on the absolute value of ω, and thus we can rewrite this as

$$\boldsymbol{F}_\alpha = -\int_{Q_\alpha} dA\,\boldsymbol{n}\int_0^{\infty}\frac{d\omega}{2\pi}(\partial_{\boldsymbol{t}\boldsymbol{t}'} + \omega^2)D(\boldsymbol{x},\boldsymbol{x},\omega) = -\int_{Q_\alpha} d\boldsymbol{A}\,p(\boldsymbol{x}), \tag{4.55}$$

where

$$p(\boldsymbol{x}) = \int_0^{\infty}\frac{d\omega}{2\pi}(\partial_{\boldsymbol{t}\boldsymbol{t}'} + \omega^2)D(\boldsymbol{x},\boldsymbol{x},\omega) \tag{4.56}$$

is interpreted as the Casimir pressure. We will now find $D(\boldsymbol{x},\boldsymbol{x},\omega)$.

4.2 The boundary integral equation

To simplify notation, we write $D(\boldsymbol{x}, \boldsymbol{x}') = D(\boldsymbol{x}, \boldsymbol{x}', \omega)$. Our approach is based on Green's second identity,

$$\int_V dV \left(\psi \nabla^2 \varphi - \varphi \nabla^2 \psi\right) = \int_{\partial V} dA \left(\psi \partial_{\boldsymbol{n}} \varphi - \varphi \partial_{\boldsymbol{n}} \psi\right). \tag{4.57}$$

This identity also holds if we replace ∇^2 with the operator $L = \nabla^2 - \omega^2$, as the terms corresponding to ω^2 will cancel on the left hand side. Let D_0 be the free Green's function of L,

$$LD_0(\boldsymbol{x}, \boldsymbol{x}'') = \delta(\boldsymbol{x} - \boldsymbol{x}''). \tag{4.58}$$

In two dimensions, this equation has the solution

$$D_0(\boldsymbol{x}, \boldsymbol{x}'') = -\frac{1}{2\pi} K_0(\omega \|\boldsymbol{x} - \boldsymbol{x}''\|) \tag{4.59}$$

where K_0 is the modified Bessel function of second kind (see Appendix B). The normal derivative on some surface with normal vector $\boldsymbol{n}$ is

$$\begin{aligned} \partial_{\boldsymbol{n}} D_0(\boldsymbol{x}, \boldsymbol{x}'') &= -\boldsymbol{n} \cdot \frac{\omega}{2\pi} K_0'(\omega \|\boldsymbol{x} - \boldsymbol{x}''\|) \nabla \|\boldsymbol{x} - \boldsymbol{x}''\| \\ &= \frac{\omega}{2\pi} K_1(\omega \|\boldsymbol{x} - \boldsymbol{x}''\|)\, \boldsymbol{n} \cdot \frac{\boldsymbol{x} - \boldsymbol{x}''}{\|\boldsymbol{x} - \boldsymbol{x}''\|}. \end{aligned} \tag{4.60}$$

Letting $\varphi = D$ and $\psi = D_0$ in (4.57) and integrating over the exterior space V_0,

$$\begin{aligned} &\int_{V_0} dV \left(D_0(\boldsymbol{x}, \boldsymbol{x}'') L D(\boldsymbol{x}, \boldsymbol{x}') - D(\boldsymbol{x}, \boldsymbol{x}') L D_0(\boldsymbol{x}, \boldsymbol{x}'')\right) \\ =& \int_{\partial V_0} dA \left(D_0(\boldsymbol{x}, \boldsymbol{x}'') \partial_{\boldsymbol{n}} D(\boldsymbol{x}, \boldsymbol{x}') - D(\boldsymbol{x}, \boldsymbol{x}') \partial_{\boldsymbol{n}} D_0(\boldsymbol{x}, \boldsymbol{x}'')\right) \\ =& \int_Q d\boldsymbol{\xi}\, D(\boldsymbol{\xi}, \boldsymbol{x}') \partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}, \boldsymbol{x}''). \end{aligned} \tag{4.61}$$

In the last step we used the fact that $\partial_{\boldsymbol{n}} D = 0$ on Q, and that $\int_{\partial V_0} = -\int_Q$.

On the other hand, if $\boldsymbol{x}', \boldsymbol{x}'' \in V_0$,

$$\begin{aligned} &\int_{V_0} dV \ \left(D_0(\boldsymbol{x}, \boldsymbol{x}'') L D(\boldsymbol{x}, \boldsymbol{x}') - D(\boldsymbol{x}, \boldsymbol{x}') L D_0(\boldsymbol{x}, \boldsymbol{x}'')\right) \\ =& \int_{V_0} dV \ \left(D_0(\boldsymbol{x}, \boldsymbol{x}'') \delta(\boldsymbol{x} - \boldsymbol{x}') - D(\boldsymbol{x}, \boldsymbol{x}') \delta(\boldsymbol{x} - \boldsymbol{x}'')\right) \\ =& D_0(\boldsymbol{x}', \boldsymbol{x}'') - D(\boldsymbol{x}'', \boldsymbol{x}'). \end{aligned} \tag{4.62}$$

Taken together, this gives

$$D(\boldsymbol{x}'', \boldsymbol{x}') = D_0(\boldsymbol{x}', \boldsymbol{x}'') - \int_Q d\boldsymbol{\xi}\, D(\boldsymbol{\xi}, \boldsymbol{x}')\partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}, \boldsymbol{x}''). \tag{4.63}$$

We are interested in calculating $D(\boldsymbol{x}', \boldsymbol{x}'')$ for $\boldsymbol{x}'$ and $\boldsymbol{x}''$ on the surfaces. When deriving (4.63), we required that $\boldsymbol{x}', \boldsymbol{x}'' \in V_0$, so in order to evaluate this on the surfaces, we must specify a limiting process.

First we shall let $\boldsymbol{x}''$ approach the surface of object α. Specifically, denote a point on Q_α by $\boldsymbol{x}_\alpha$, and let $\boldsymbol{x}'' \to \boldsymbol{x}_\alpha$. The emerging issue is that $\partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}_\alpha, \boldsymbol{x}_\alpha)$ is singular at $\boldsymbol{\xi}_\alpha = \boldsymbol{x}_\alpha$. There are no problems on other surfaces, so we isolate the issue by writing

$$\begin{aligned} D(\boldsymbol{x}_\alpha, \boldsymbol{x}') = D_0(\boldsymbol{x}', \boldsymbol{x}_\alpha) &- \sum_{\gamma \neq \alpha} \int_{Q_\gamma} d\boldsymbol{\xi}_\gamma\, D(\boldsymbol{\xi}_\gamma, \boldsymbol{x}')\partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}_\gamma, \boldsymbol{x}_\alpha) \\ &- \lim_{\boldsymbol{x}'' \to \boldsymbol{x}_\alpha} \int_{Q_\alpha} d\boldsymbol{\xi}_\alpha\, D(\boldsymbol{\xi}_\alpha, \boldsymbol{x}')\partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}_\alpha, \boldsymbol{x}''). \end{aligned} \tag{4.64}$$

In order to evaluate the second integral, we first split it into two parts,

$$\int_{Q_\alpha} d\boldsymbol{\xi}_\alpha\, D(\boldsymbol{\xi}_\alpha, \boldsymbol{x}')\partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}_\alpha, \boldsymbol{x}'') = \left[\int_{S_\varepsilon} + \int_{C_\varepsilon}\right] d\boldsymbol{\xi}_\alpha\, D(\boldsymbol{\xi}_\alpha, \boldsymbol{x}')\partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}_\alpha, \boldsymbol{x}'') \tag{4.65}$$

where C_ε is a small section around $\boldsymbol{x}_\alpha$ with radius ε, and S_ε is the remainder of the curve. As $\varepsilon \to 0$, the part over S_ε becomes a principal value integral. For the part over C_ε, we start by assuming that D is approximately constant on the interval and can thus be brought outside the integral:

$$\int_{C_\varepsilon} d\boldsymbol{\xi}_\alpha\, D(\boldsymbol{\xi}_\alpha, \boldsymbol{x}')\partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}_\alpha, \boldsymbol{x}'') \approx D(\boldsymbol{x}_\alpha, \boldsymbol{x}') \int_{C_\varepsilon} d\boldsymbol{\xi}_\alpha\, \partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}_\alpha, \boldsymbol{x}''). \tag{4.66}$$

Inserting this in (4.64) gives us the equation

$$\begin{aligned} D(\boldsymbol{x}_\alpha, \boldsymbol{x}') = D_0(\boldsymbol{x}', \boldsymbol{x}_\alpha) &- \sum_{\gamma} PV_{\boldsymbol{x}_\alpha} \int_{Q_\gamma} d\boldsymbol{\xi}_\gamma\, D(\boldsymbol{\xi}_\alpha, \boldsymbol{x}')\partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}_\alpha, \boldsymbol{x}_\alpha) \\ &- D(\boldsymbol{x}_\alpha, \boldsymbol{x}') \lim_{\varepsilon \to 0} \int_{C_\varepsilon} d\boldsymbol{\xi}_\alpha\, \partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}_\alpha, \boldsymbol{x}_\alpha). \end{aligned} \tag{4.67}$$

Let[†]

$$E(\boldsymbol{x},\boldsymbol{x}') = \partial_{\boldsymbol{n}'} D_0(\boldsymbol{x}',\boldsymbol{x}) = -\frac{\omega}{2\pi} K_1(\omega\|\boldsymbol{x}-\boldsymbol{x}'\|)\ \boldsymbol{n}'\cdot\frac{\boldsymbol{x}-\boldsymbol{x}'}{\|\boldsymbol{x}-\boldsymbol{x}'\|} \tag{4.68}$$

where $\boldsymbol{n}' = \boldsymbol{n}(\boldsymbol{x}')$, and write (4.67) as

$$\kappa D(\boldsymbol{x}_\alpha,\boldsymbol{x}') = D_0(\boldsymbol{x}_\alpha,\boldsymbol{x}') - \sum_\gamma PV_{\boldsymbol{x}_\alpha}\int_Q d\boldsymbol{\xi}\, E(\boldsymbol{x}_\alpha,\boldsymbol{\xi}) D(\boldsymbol{\xi},\boldsymbol{x}'), \tag{4.69}$$

where the principal value becomes relevant only when $\gamma = \alpha$, and

$$\kappa = 1 + \lim_{\varepsilon\to 0}\int_{C_\varepsilon} d\boldsymbol{\xi}_\alpha\, \partial_{\boldsymbol{n}} D_0(\boldsymbol{\xi}_\alpha,\boldsymbol{x}_\alpha). \tag{4.70}$$

The evaluation of this integral is an interesting point. In the Dirichlet case, a corresponding integral was evaluated to zero using the technique of changing the integral path to move along small semicircle around $\boldsymbol{x}_\alpha$ [14] [20]. In that case, the integrand was D_0 instead of $\partial_{\boldsymbol{n}} D_0$, which has a weaker singularity. If we use the same regularization technique here we find $\kappa = 1/2$. However, another technique gives $\kappa = 1$, and in fact we found this to yield numerical answers that matched the exact values. This point is discussed in much detail in Section 5.3. For now, let us keep κ as an unknown constant.

Moving on, the next step is to let $\boldsymbol{x}'$ approach the surface of object β. For $\beta \neq \alpha$, this is no problem. However, in the limit $\boldsymbol{x}_\beta \to \boldsymbol{x}_\alpha$, D_0 blows up. We approach this problem by introducing the function D_α, representing the Green's function if α had been the only object. According to (4.69) the equation for D_α must be

$$\kappa D_\alpha(\boldsymbol{x}_\alpha,\boldsymbol{x}') = D_0(\boldsymbol{x}_\alpha,\boldsymbol{x}') - PV_{\boldsymbol{x}_\alpha}\int_{Q_\alpha} d\boldsymbol{\xi}_\alpha\, E(\boldsymbol{x}_\alpha,\boldsymbol{\xi}_\alpha) D_\alpha(\boldsymbol{\xi}_\alpha,\boldsymbol{x}'). \tag{4.71}$$

We shall refer to D_α as the self-pressure. Next, let

$$P(\boldsymbol{x}_\alpha,\boldsymbol{x}'_\beta) = D(\boldsymbol{x}_\alpha,\boldsymbol{x}'_\beta) - \delta_{\alpha\beta} D_\beta(\boldsymbol{x}_\alpha,\boldsymbol{x}'_\beta). \tag{4.72}$$

Importantly, for the case $\alpha = \beta$, P does not have the aforementioned singularity from D_0. Furthermore, we claim that if we replace D by P in (4.55), the

[†]Be aware that order of the arguments are swapped here. This will simplify notation later. The arguments of D_0 are swapped as well in (4.69), but this does not change its value.

contribution from D_α must be zero. Otherwise, if α was truly the only object, there would be a nonzero net force on it, which would cause an acceleration that contradicts conservation of momentum.

There still appears to be a problem in (4.71), namely that $D_0(\boldsymbol{x}_\alpha, \boldsymbol{x}')$ still blows up around $\boldsymbol{x}' = \boldsymbol{x}_\alpha$. This means that D_α might exhibit singular behaviour here. However, D_α only appears inside an integral, so this will not be a problem if the singularity is integrable. We shall come back to this point in Section 4.4, where we argue that it is indeed integrable. Until then, this issue will not invalidate the rest of our discussion.

For $\alpha \neq \beta$, inserting (4.72) in (4.69) gives

$$\begin{aligned}\kappa P(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) &= D_0(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) - PV_{\boldsymbol{x}_\alpha} \int_Q d\boldsymbol{\xi}\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}) D(\boldsymbol{\xi}, \boldsymbol{x}_\beta) \\ &= D_0(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) - \int_{Q_\beta} d\boldsymbol{\xi}_\beta\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}_\beta) D(\boldsymbol{\xi}_\beta, \boldsymbol{x}_\beta) \\ &\qquad - \sum_{\gamma \neq \beta} PV_{\boldsymbol{x}_\alpha} \int_{Q_\gamma} d\boldsymbol{\xi}_\gamma\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}_\gamma) P(\boldsymbol{\xi}_\gamma, \boldsymbol{x}_\beta).\end{aligned} \tag{4.73}$$

But $D(\boldsymbol{\xi}_\beta, \boldsymbol{x}_\beta) = D_\beta(\boldsymbol{x}_\beta, \boldsymbol{x}'_\beta) + P(\boldsymbol{\xi}_\beta, \boldsymbol{x}_\beta)$, so

$$\begin{aligned}\kappa P(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) = D_0(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) &- \int_{Q_\beta} d\boldsymbol{\xi}_\beta\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}_\beta) D_\beta(\boldsymbol{\xi}_\beta, \boldsymbol{x}_\beta) \\ &- \sum_{\gamma} PV_{\boldsymbol{x}_\alpha} \int_{Q_\gamma} d\boldsymbol{\xi}_\gamma\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}_\gamma) P(\boldsymbol{\xi}_\gamma, \boldsymbol{x}_\beta).\end{aligned} \tag{4.74}$$

For $\alpha = \beta$, we have

$$\begin{aligned}\kappa P(\boldsymbol{x}_\alpha, \boldsymbol{x}'_\alpha) &= D_0(\boldsymbol{x}_\alpha, \boldsymbol{x}'_\alpha) - PV_{\boldsymbol{x}_\alpha} \int_Q d\boldsymbol{\xi}\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}) D(\boldsymbol{\xi}, \boldsymbol{x}'_\alpha) \\ &\qquad - \left[D_0(\boldsymbol{x}_\alpha, \boldsymbol{x}'_\alpha) - PV_{\boldsymbol{x}_\alpha} \int_{Q_\alpha} d\boldsymbol{\xi}_\alpha\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}_\alpha) D_\alpha(\boldsymbol{\xi}_\alpha, \boldsymbol{x}'_\alpha) \right] \\ &= - \sum_{\gamma \neq \alpha} \int_{Q_\gamma} d\boldsymbol{\xi}_\gamma\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}_\gamma) P(\boldsymbol{\xi}_\gamma, \boldsymbol{x}_\alpha) \\ &\qquad - \int_{Q_\alpha} d\boldsymbol{\xi}_\alpha\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}_\alpha) \left(D(\boldsymbol{\xi}_\alpha, \boldsymbol{x}'_\alpha) - D_\alpha(\boldsymbol{\xi}_\alpha, \boldsymbol{x}'_\alpha) \right) \\ &= - \sum_{\gamma} PV_{\boldsymbol{x}_\alpha} \int_{Q_\gamma} d\boldsymbol{\xi}_\gamma\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}_\gamma) P(\boldsymbol{\xi}_\gamma, \boldsymbol{x}_\alpha).\end{aligned} \tag{4.75}$$

Then for general α and β, we can write

$$\kappa P(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) = V(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) - PV_{\boldsymbol{x}_\alpha} \int_Q d\boldsymbol{\xi}\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}) P(\boldsymbol{\xi}, \boldsymbol{x}_\beta), \tag{4.76}$$

where

$$V(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) = D_0(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) - \int_{Q_\beta} d\boldsymbol{\xi}_\beta\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}_\beta) D_\beta(\boldsymbol{\xi}_\beta, \boldsymbol{x}_\beta) \tag{4.77}$$

when $\alpha \neq \beta$ and zero otherwise.

4.3 Discretization

We will now discretize the surfaces in order to turn the integral equation into a set of linear equations.

We approximate each surface as piecewise linear curve. The surface Q_α is divided into line segments I_α^i for $i = 1, \ldots, N$. Let $\boldsymbol{x}_\alpha^i$ be the midpoint on I_α^i and let L_α^i be the length of that segment. We can then split the integral in (4.76) as

$$\int_Q d\boldsymbol{\xi}\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}) P(\boldsymbol{\xi}, \boldsymbol{x}_\beta) = \sum_\gamma \sum_{k=1}^{N} \int_{I_\gamma^k} d\boldsymbol{\xi}_\gamma\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}_\gamma) P(\boldsymbol{\xi}_\gamma, \boldsymbol{x}_\beta). \tag{4.78}$$

Assuming that $P(\boldsymbol{\xi}_\gamma, \boldsymbol{x}_\beta)$ varies little on I_γ^k, it can be brought outside the integral and evaluated at $\boldsymbol{\xi}_\gamma = \boldsymbol{x}_\gamma^k$:

$$\int_{I_\gamma^k} d\boldsymbol{\xi}_\gamma\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}_\gamma) P(\boldsymbol{\xi}_\gamma, \boldsymbol{x}_\beta) = P(\boldsymbol{x}_\gamma^k, \boldsymbol{x}_\beta) \int_{I_\gamma^k} d\boldsymbol{\xi}_\gamma\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}_\alpha). \tag{4.79}$$

We will calculate the pressure at the midpoints, i.e. at $\boldsymbol{x}_\alpha = \boldsymbol{x}_\alpha^i$ and $\boldsymbol{x}_\beta = \boldsymbol{x}_\beta^j$. Let $P_{\alpha\beta}^{ij} = P(\boldsymbol{x}_\alpha^i, \boldsymbol{x}_\beta^j)$, and similarly define

$$V_{\alpha\beta}^{ij} = D_0(\boldsymbol{x}_\alpha^i, \boldsymbol{x}_\beta^j) - \int_{Q_\beta} d\boldsymbol{\xi}_\beta\, E(\boldsymbol{x}_\alpha^i, \boldsymbol{\xi}_\beta) D_\beta(\boldsymbol{\xi}_\beta, \boldsymbol{x}_\beta^j). \tag{4.80}$$

In this manner we discretize (4.76), turning it into the set of equations

$$\kappa P_{\alpha\beta}^{ij} = V_{\alpha\beta}^{ij} - \sum_\gamma \sum_{k=1}^{N} P_{\gamma\beta}^{kj}\, PV_{\boldsymbol{x}_\alpha^i} \int_{I_\gamma^k} d\boldsymbol{\xi}_\gamma\, E(\boldsymbol{x}_\alpha^i, \boldsymbol{\xi}_\gamma). \tag{4.81}$$

Since $E(\boldsymbol{x}_\alpha^i, \boldsymbol{\xi}_\gamma)$ contains a factor on the form $\boldsymbol{n}(\boldsymbol{\xi}_\gamma) \cdot (\boldsymbol{x}_\alpha^i - \boldsymbol{\xi}_\gamma)$, the integrand vanishes on I_α^i where this factor is zero. The other integrals are well-behaved, and can be evaluated for example with the midpoint rule,

$$\int_{I_\gamma^k} d\boldsymbol{\xi}_\gamma \, E(\boldsymbol{x}_\alpha^i, \boldsymbol{\xi}_\gamma) \approx L_\gamma^k E(\boldsymbol{x}_\alpha^i, \boldsymbol{x}_\gamma^k). \tag{4.82}$$

Finally, if we let

$$M_{\alpha\gamma}^{ik} = \kappa \delta_{\alpha\gamma} \delta_{ik} + (1 - \delta_{\alpha\gamma}\delta_{ik}) L_\gamma^k E(\boldsymbol{x}_\alpha^i, \boldsymbol{x}_\gamma^k), \tag{4.83}$$

we can write this set of equations as

$$\sum_\gamma \sum_{k=1}^N M_{\alpha\gamma}^{ik} P_{\gamma\beta}^{kj} = V_{\alpha\beta}^{ij}. \tag{4.84}$$

This discretizes our equation into a system of equations. For fixed β and j, this represents a set of Nr linear equations.

To write it even more compactly, we can let P, V and M be block matrices with entries $P_{\alpha\beta}^{ij}$, $V_{\alpha\beta}^{ij}$ and $M_{\alpha\beta}^{ij}$ respectively (see (1.7)), for $\alpha, \beta = 1, \ldots, r$ and $i, j = 1, \ldots, N$. Then (4.84) can be written compactly as

$$MP = V. \tag{4.85}$$

Each column in this matrix equation represents a set of Nr equations and unknowns. Thus, to solve for the Green's function at $\boldsymbol{x}_\beta^j$, we must first solve

$$M\boldsymbol{p}(\boldsymbol{x}_\beta^j) = \boldsymbol{v}(\boldsymbol{x}_\beta^j), \tag{4.86}$$

where $\boldsymbol{p}(\boldsymbol{x}_\beta^j)$ and $\boldsymbol{v}(\boldsymbol{x}_\beta^j)$ are block vectors with entries $P(\boldsymbol{x}_\alpha^i, \boldsymbol{x}_\beta^j)$ and $V(\boldsymbol{x}_\alpha^i, \boldsymbol{x}_\beta^j)$ respectively, for $\alpha = 1, \ldots, r$ and $i = 1, \ldots, N$. In finding the Casimir pressure, we are especially interested in the entry $P_{\beta\beta}^{jj} = P(\boldsymbol{x}_\beta^j, \boldsymbol{x}_\beta^j)$ appearing in (4.56).

In the pressure integral (4.56) we have a term $\partial_{\boldsymbol{t}} \partial_{\boldsymbol{t}'} P(\boldsymbol{x}, \boldsymbol{x}, \omega)$, where the $\partial_{\boldsymbol{t}'}$ indicates that it should be applied to the second argument. In discretizing, we must also discretize the operator. To develop a finite difference formula for this expression, we start from Taylor's theorem,

$$\begin{aligned} f(\boldsymbol{x} + h_+ \boldsymbol{t}) &= f(\boldsymbol{x}) + h_+ \partial_{\boldsymbol{t}} f(\boldsymbol{x}) + \mathcal{O}(h_+^2), \\ f(\boldsymbol{x} - h_- \boldsymbol{t}) &= f(\boldsymbol{x}) - h_- \partial_{\boldsymbol{t}} f(\boldsymbol{x}) + \mathcal{O}(h_-^2). \end{aligned} \tag{4.87}$$

Subtracting these, we get

$$\partial_{\boldsymbol{t}} f(\boldsymbol{x}) \approx \frac{f(\boldsymbol{x}+h_+) - f(\boldsymbol{x}-h_-)}{h_+ + h_-}. \tag{4.88}$$

In our particular situation, we have $\boldsymbol{x}_\alpha^j \pm h_\pm \boldsymbol{t} \approx \boldsymbol{x}_\alpha^{j\pm1}$ when $h_\pm$ is the distance between $\boldsymbol{x}_\alpha^j$ and $\boldsymbol{x}_\alpha^{j\pm1}$. Applying this, we get

$$\partial_{\boldsymbol{t}} P_{\alpha\beta}(\boldsymbol{x}_\alpha^i, \boldsymbol{x}_\beta^j) \approx \frac{P_{\alpha\beta}(\boldsymbol{x}_\alpha^{i+1}, \boldsymbol{x}_\beta^j) - P_{\alpha\beta}(\boldsymbol{x}_\alpha^{i-1}, \boldsymbol{x}_\beta^j)}{\|\boldsymbol{x}_\alpha^{i+1} - \boldsymbol{x}_\alpha^i\| + \|\boldsymbol{x}_\alpha^i - \boldsymbol{x}_\alpha^{i-1}\|} = \frac{P_{\alpha\beta}^{i+1,j} - P_{\alpha\beta}^{i-1,j}}{h_+ + h_-}. \tag{4.89}$$

Next we apply $\partial_{t'}$ to this expression:

$$\partial_{tt'} P_{\alpha\beta}(\boldsymbol{x}_\alpha^i, \boldsymbol{x}_\beta^j) \approx \Delta^{ij} P_{\alpha\beta}^{ij} = \frac{P_{\alpha\beta}^{i+1,j+1} - P_{\alpha\beta}^{i+1,j-1} - P_{\alpha\beta}^{i-1,j+1} + P_{\alpha\beta}^{i-1,j-1}}{(h_+ + h_-)(h'_+ + h'_-)}. \tag{4.90}$$

For shorthand, we have introduced the discrete derivative operator Δ^{ij}, which can be defined as above or in any other convenient way.

At last the pressure from (4.56) can be written as

$$p(\boldsymbol{x}_\beta^j) = \int_0^\infty \frac{d\omega}{2\pi} (\Delta^{jj} + \omega^2) P_{\beta\beta}^{jj}, \tag{4.91}$$

where $P_{\beta\beta}^{jj}$ implicitly is a function of ω. While the ω integral is not actually discretized, we assume $P_{\beta\beta}^{jj}$ as a function of ω is finite, continuous, and converges to 0 sufficiently fast. Thus, the integral is straightforward to implement numerically, for example by selecting a cutoff (which might have to be done a posteriori, after sampling $P_{\beta\beta}^{jj}$ for a few values of ω) and using a Gaussian quadrature.

4.4 The self-pressure

In order to calculate the source (4.80), we need to evaluate the integral

$$\int_{Q_\beta} d\boldsymbol{\xi}_\beta \, E(\boldsymbol{x}_\alpha^i, \boldsymbol{\xi}_\beta) D_\beta(\boldsymbol{\xi}_\beta, \boldsymbol{x}_\beta^j). \tag{4.92}$$

Assuming D_β is well-behaved, we split this integral, then approximate it using the midpoint rule:

$$\begin{aligned}\int_{Q_\beta} d\boldsymbol{\xi}_\beta\, E(\boldsymbol{x}_\alpha^i, \boldsymbol{\xi}_\beta) D_\beta(\boldsymbol{\xi}_\beta, \boldsymbol{x}_\beta^j) &= \sum_{k=1}^{N} \int_{I_\beta^k} d\boldsymbol{\xi}_\beta\, E(\boldsymbol{x}_\alpha^i, \boldsymbol{\xi}_\beta) D_\beta(\boldsymbol{\xi}_\beta, \boldsymbol{x}_\beta^j) \\ &\approx \sum_{k=1}^{N} E(\boldsymbol{x}_\alpha^i, \boldsymbol{x}_\beta^k) D_\beta(\boldsymbol{x}_\beta^k, \boldsymbol{x}_\beta^j)\end{aligned} \tag{4.93}$$

and (4.80) turns into

$$V_{\alpha\beta}^{ij} = D_0(\boldsymbol{x}_\alpha^i, \boldsymbol{x}_\beta^j) - \sum_{k=1}^{N} E(\boldsymbol{x}_\alpha^i, \boldsymbol{x}_\beta^k) D_\beta(\boldsymbol{x}_\beta^k, \boldsymbol{x}_\beta^j) \tag{4.94}$$

This set of equations for a fixed $\boldsymbol{x}_\beta^j$ can be written as

$$\boldsymbol{v}_\alpha(\boldsymbol{x}_\beta^j) = \boldsymbol{D}_{0,\alpha}(\boldsymbol{x}_\beta^j) - M_{\alpha\beta} \boldsymbol{D}_\beta(\boldsymbol{x}_\beta^j), \tag{4.95}$$

where $\boldsymbol{v}_\alpha(\boldsymbol{x}_\beta^j)$ is the vector with entries $V(\boldsymbol{x}_\alpha^i, \boldsymbol{x}_\beta^j)$ for $i = 1, \ldots, N$, and we define $\boldsymbol{D}_{0,\alpha}(\boldsymbol{x}_\beta^j)$ and $\boldsymbol{D}_\beta(\boldsymbol{x}_\beta^j)$ similarly.

This means we need to find $D_\beta(\boldsymbol{x}_\beta^k, \boldsymbol{x}_\beta^j)$ for $k = 1, \ldots, N$. (4.71) gives us the equation

$$\kappa D_\beta(\boldsymbol{x}_\beta^k, \boldsymbol{x}_\beta^j) = D_0(\boldsymbol{x}_\beta^k, \boldsymbol{x}_\beta^j) - PV_{\boldsymbol{x}_\beta^k} \int_{Q_\beta} d\boldsymbol{\xi}_\beta\, E(\boldsymbol{x}_\beta^k, \boldsymbol{\xi}_\beta) D_\beta(\boldsymbol{\xi}_\beta, \boldsymbol{x}_\beta^j). \tag{4.96}$$

This immediately presents a problem for $k = j$, as D_0 has a singularity when $\boldsymbol{x}_\beta^k = \boldsymbol{x}_\beta^j$. Since (4.96) is linear and the asymptotic behaviour of D_0 is $D_0 \sim \log \varepsilon$ (see (B.12)), we expect that D_β also has a singularity proportional to $\log \varepsilon$. This singularity is integrable, and we can evaluate D_β at $\boldsymbol{x}'$ close to $\boldsymbol{x}_\beta^j$, e.g. at $\boldsymbol{x}' = \boldsymbol{x}_\beta^j + \varepsilon \boldsymbol{n}(\boldsymbol{x}_\beta^j)$. Then, when we insert $D_\beta(\boldsymbol{x}_\beta^k, \boldsymbol{x}')$ in (4.80), we expect to see that the integral still converges and is independent of ε as long as it is small.

Additionally, the fact that the the magnitude of D_β is similar to the magnitude of D_0, both in maximum value and in shape, has another implication. When we discretize (4.96), $D_\beta(\boldsymbol{\xi}_\beta, \boldsymbol{x}')$ will have a peak around $\boldsymbol{\xi}_\beta = \boldsymbol{x}'$ which is proportional to the peak of $D_0(\boldsymbol{x}, \boldsymbol{x} + \varepsilon \boldsymbol{n})$. In order to get a good approximation that allows us to apply the midpoint rule, we must must have a sufficiently good resolution of the curve that accurately resolves this peak.

For a sufficiently large N, we split the integral and bring D_β outside:

$$\kappa D_\beta(\boldsymbol{x}_\beta^k, \boldsymbol{x}') = D_0(\boldsymbol{x}_\beta^k, \boldsymbol{x}') - \sum_{l=1}^{N} D_\beta(\boldsymbol{x}_\beta^l, \boldsymbol{x}') PV_{\boldsymbol{x}_\beta^k} \int_{I_\beta^l} d\boldsymbol{\xi}_\beta \, E(\boldsymbol{x}_\beta^k, \boldsymbol{\xi}_\beta). \quad (4.97)$$

Vectorizing as before gives us

$$M_{\beta\beta} \boldsymbol{D}_\beta(\boldsymbol{x}') = \boldsymbol{D}_{0,\beta}(\boldsymbol{x}'). \quad (4.98)$$

where the entries of the matrix $M_{\beta\beta}$ are

$$M_{\beta\beta}^{kl} = \kappa\delta_{kl} + (1 - \delta_{kl}) L_\beta^l E(\boldsymbol{x}_\beta^k, \boldsymbol{x}_\beta^l). \quad (4.99)$$

Solving this for $\boldsymbol{D}_\beta(\boldsymbol{x}')$ at some $\boldsymbol{x}'$ close to $\boldsymbol{x}_\beta^j$, we can insert it in (4.95) and get the source term $\boldsymbol{v}_\alpha(\boldsymbol{x}_\beta^j)$.

4.5 Example: parallel plates

We now consider a specific case of two parallel plates. Because it is impossible to treat infinite plates numerically, we make the approximation that there are two plates of length L at a distance a, where $L \gg a$. Near the middle of these plates, the situation is similar to the one where the plates are infinite, and we shall view the pressure at this point as representative for the pressure everywhere on the infinite plates.

Choosing an equidistant discretization, the length of each segment is $L_\beta^j = L/N$. We need to calculate the matrix elements $M_{\alpha\beta}^{ij}$ and v_α^i in (4.86). Let's start with $M_{\alpha\beta}^{ij}$. Since $E_{\alpha\beta}^{ij}$ contains a factor on the form $\boldsymbol{n}_\beta \cdot \left(\boldsymbol{x}_\alpha^i - \boldsymbol{x}_\beta^j\right)$, we immediately see that $M_{\alpha\beta}^{ij} = \kappa\delta_{ij}$ when $\alpha = \beta$, as $\boldsymbol{x}_\alpha^i$ and $\boldsymbol{x}_\beta^j$ both lie on the same plate and their difference is perpendicular to $\boldsymbol{n}$. For $\alpha \neq \beta$, $M_{\alpha\beta}^{ij}$ is readily computable. Let $d_{ij} = \frac{i-j}{N} L$ be the parallel distance between $\boldsymbol{x}_\alpha^i$ and $\boldsymbol{x}_\beta^j$. With the perpendicular distance between the two plates being a, we have

$$\boldsymbol{x}_\alpha^i - \boldsymbol{x}_\beta^j = a\boldsymbol{n}_\beta + d_{ij}\boldsymbol{t}_\beta. \quad (4.100)$$

Then (5.8) gives

$$M^{ij}_{\alpha\beta} = L^j_\beta E(\boldsymbol{x}^i_\alpha, \boldsymbol{x}^j_\beta) = -\frac{L}{N}\frac{\omega}{2\pi} K_1(\omega\|\boldsymbol{x}^i_\alpha - \boldsymbol{x}^j_\beta\|)\, \boldsymbol{n}_\beta \cdot \frac{\boldsymbol{x}^i_\alpha - \boldsymbol{x}^j_\beta}{\|\boldsymbol{x}^i_\alpha - \boldsymbol{x}^j_\beta\|} \\ = -\frac{aL\omega}{2\pi N}\frac{K_1\left(\omega\sqrt{a^2+d^2_{ij}}\right)}{\sqrt{a^2+d^2_{ij}}} \quad . \tag{4.101}$$

Next, let us look at $\boldsymbol{v}^{\ j}_{\alpha\beta}$. Recalling that the source vanishes for $\alpha = \beta$, so the first term in (4.95) becomes,

$$D_0(\boldsymbol{x}^i_\alpha, \boldsymbol{x}^j_\beta) = -\frac{1}{2\pi} K_0\left(\omega\sqrt{a^2+d^2_{ij}}\right) \tag{4.102}$$

is well-behaved. The self pressure is given by

$$M_{\beta\beta}\boldsymbol{D}^j_\beta = \boldsymbol{D}_{0,\beta}(\boldsymbol{x}'), \tag{4.103}$$

but since $M_{\beta\beta} = \kappa I$, we simply have $\boldsymbol{D}^j_\beta = \frac{1}{\kappa}\boldsymbol{D}_{0,\beta}(\boldsymbol{x}')$.

At this point, all the unknowns in (4.86) are defined, and we can solve the system to get $P^{ij}_{\alpha\beta}$. To approximate the situation for infinite plates, we will evaluate this at the middle, i.e. at $\boldsymbol{x} = \boldsymbol{x}^{\frac{N}{2}}_\alpha$. Since plates are translationally invariant, $\partial_{\boldsymbol{tt}'} D = 0$. Then, according to (4.56), the pressure is given by

$$p(\boldsymbol{x}^{\frac{N}{2}}_\alpha) = \int_0^\infty \frac{d\omega}{2\pi}\omega^2 P^{\frac{N}{2}\frac{N}{2}}_{\beta\beta}. \tag{4.104}$$

Figure 4.1 shows the calculated pressure p as a function of a, together with the exact pressure,

$$p_{exact} = -\frac{\zeta(3)}{8\pi a^3}. \tag{4.105}$$

The X'es indicates the points computed by BIM multiplied by a factor 2, and match the exact solution well. Apart from this factor, the geometry dependence is correct. Kilen and Mikalsen found this same factor was found to be missing for Dirichlet cases in both 2 and 3 dimensions [14] [20]. This observation suggests the factor is also missing for von Neumann boundary conditions. This point will be discussed in depth in Section 5.4.

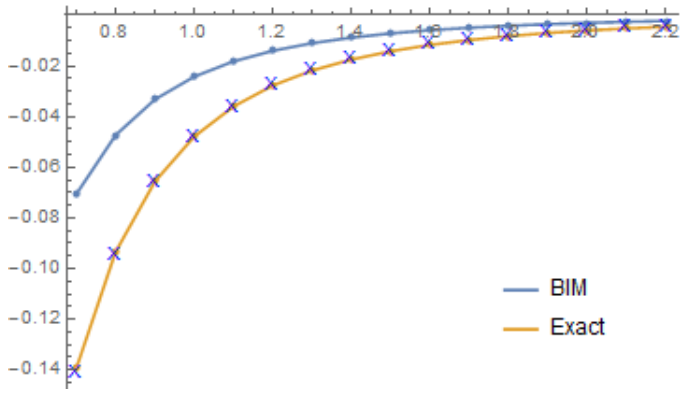


Figure 4.1: The Casimir pressure between two parallel plates at a separation distance a as computed by BIM, together with the exact value from mode expansion. The X'es mark values computed by BIM multiplied by 2, which match the exact solution. The functions were sampled for $0.7 < a < 2.2$ with steps 0.1. The plates from BIM were approximated by finite plates of length $L = 10$. The resolution used was $N = 100$, and the integral over ω was computed using Simpson's rule with 20 points.

4.6 Example: concentric circles

Consider a specific case of two concentric circles Q_1 and Q_2 with radii r_1 and r_2, with $r_1 < r_2$.

To calculate the pressure numerically, discretize the circles according to

$$\boldsymbol{x}_\beta^j = \begin{bmatrix} r_\beta \cos(2\pi j/N) \\ r_\beta \sin(2\pi j/N) \end{bmatrix}. \tag{4.106}$$

The normals point into the space between the circles,

$$\boldsymbol{n}_1^j = \begin{bmatrix} \cos(2\pi j/N) \\ \sin(2\pi j/N) \end{bmatrix}, \qquad \boldsymbol{n}_2^j = -\begin{bmatrix} \cos(2\pi j/N) \\ \sin(2\pi j/N) \end{bmatrix}. \tag{4.107}$$

Using these parametrizations, we can find $M_{\alpha\beta}^{ij}$ and $\boldsymbol{v}_\beta^j$ according to (4.83) and (4.95) and solve (4.86). Due to rotational symmetry, the pressure will be the same on the whole circle. Thus, we only need to calculate the pressure at one point on each circle, and moreover, $\partial_{\boldsymbol{tt}'} D = 0$ in (4.56).

The numerical calculations performed by the author did not give an answer that matched the exact pressure given in (2.155). This worrisome fact is discussed more closely in Section 5.2.

4.7 Symmetry reductions

Lastly we will look at how symmetries can be used to reduce the number of points required for the computations. We already used symmetries in our examples of parallel plates and concentric circles to reduce the number of points where the pressure must be calculated. In those cases we used ad hoc arguments, and we now generalise that notion.

Let h be an isometry that preserves Q. An isometry means that all lengths are invariant under the action of this function, i.e. $\|\boldsymbol{x} - \boldsymbol{x}'\| = \|h\boldsymbol{x} - h\boldsymbol{x}'\|$ for all $\boldsymbol{x}$ and $\boldsymbol{x}'$. The isometries in $\mathbb{R}^n$ are translations, rotations, reflections, and compositions of these. The isometries form a group under composition.

When we say it preserves Q, we mean that $h(Q) = Q$. This means that h essentially permutes the surfaces $Q_1, \ldots, Q_r$, or in other words, for every α there is an α' such that

$$h(Q_\alpha) = Q_{\alpha'}. \tag{4.108}$$

Throughout, we shall use this convention that α' is the index of the surface such that $h(Q_\alpha) = Q_{\alpha'}$. If two objects are related through an isometry, we say that the two objects are congruent.

Recall that the equation for P is

$$\kappa P(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) = V(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) - PV_{\boldsymbol{x}_\alpha} \int_Q d\boldsymbol{\xi}\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}) P(\boldsymbol{\xi}, \boldsymbol{x}_\beta). \tag{4.109}$$

What we wish to show is that

$$P(h\boldsymbol{x}, \boldsymbol{x}') = P(\boldsymbol{x}, h^{-1}\boldsymbol{x}'). \tag{4.110}$$

To do this, let $R(\boldsymbol{x}, \boldsymbol{x}') = P(h\boldsymbol{x}, h\boldsymbol{x}')$. If $R(\boldsymbol{x}, \boldsymbol{x}') = P(\boldsymbol{x}, \boldsymbol{x}')$, it implies (4.110). We shall prove this by showing that

$$\kappa R(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) = V(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) - PV_{\boldsymbol{x}_\alpha} \int_Q d\boldsymbol{\xi}\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}) R(\boldsymbol{\xi}, \boldsymbol{x}_\beta), \tag{4.111}$$

i.e. that R solves (4.109), entailing $P = R$ by uniqueness.

Isometries preserve norms and scalar products, so

$$D_0(\boldsymbol{x}, \boldsymbol{x}') = D_0(h\boldsymbol{x}, h\boldsymbol{x}'), \tag{4.112}$$

$$E(\boldsymbol{x}, \boldsymbol{x}') = E(h\boldsymbol{x}, h\boldsymbol{x}'), \tag{4.113}$$

because they only depend on $\|\boldsymbol{x}-\boldsymbol{x}'\|$ and $\boldsymbol{n}\cdot(\boldsymbol{x}-\boldsymbol{x}')$. Processing the integral, first substitute $\boldsymbol{\xi}'=h\boldsymbol{\xi}$. Since h is an isometry, it preserves areas so $d\boldsymbol{\xi}'=d\boldsymbol{\xi}$, and the integration domain Q is invariant under h. The principal value of the new integral should be taken at $h\boldsymbol{x}_\alpha$. Then

$$\begin{aligned} PV_{\boldsymbol{x}_\alpha}\int_Q d\boldsymbol{\xi}\, E(\boldsymbol{x}_\alpha,\boldsymbol{\xi})R(\boldsymbol{\xi},\boldsymbol{x}_\beta) &= PV_{\boldsymbol{x}_\alpha}\int_Q d\boldsymbol{\xi}\, E(\boldsymbol{x}_\alpha,\boldsymbol{\xi})P(h\boldsymbol{\xi},h\boldsymbol{x}_\beta) \\ &= PV_{h\boldsymbol{x}_\alpha}\int_Q d\boldsymbol{\xi}'\, E(\boldsymbol{x}_\alpha,h^{-1}\boldsymbol{\xi}')P(\boldsymbol{\xi}',h\boldsymbol{x}_\beta) \\ &= PV_{h\boldsymbol{x}_\alpha}\int_Q d\boldsymbol{\xi}'\, E(h\boldsymbol{x}_\alpha,\boldsymbol{\xi}')P(\boldsymbol{\xi}',h\boldsymbol{x}_\beta). \end{aligned} \tag{4.114}$$

This same way of transforming the integral will be used several times in the subsequent discussion. Next, consider

$$V(\boldsymbol{x}_\alpha,\boldsymbol{x}_\beta)=D_0(\boldsymbol{x}_\alpha,\boldsymbol{x}_\beta)-\int_{Q_\beta}d\boldsymbol{\xi}_\beta\, E(\boldsymbol{x}_\alpha,\boldsymbol{\xi}_\beta)D_\beta(\boldsymbol{\xi}_\beta,\boldsymbol{x}_\beta). \tag{4.115}$$

As mentioned, D_0 and E are invariant under h. The self-pressure is defined by the equation

$$\kappa D_\beta(\boldsymbol{x}_\beta,\boldsymbol{x}')=D_0(\boldsymbol{x}_\beta,\boldsymbol{x}')-PV_{\boldsymbol{x}_\beta}\int_{Q_\beta}d\boldsymbol{\xi}_\beta\, E(\boldsymbol{x}_\beta,\boldsymbol{\xi}_\beta)D_\beta(\boldsymbol{\xi}_\beta,\boldsymbol{x}'). \tag{4.116}$$

Note that

$$\begin{aligned} \kappa D_{\beta'}(h\boldsymbol{x}_\beta,h\boldsymbol{x}') &= D_0(h\boldsymbol{x}_\beta,h\boldsymbol{x}')-PV_{h\boldsymbol{x}_\beta}\int_{Q'_\beta}d\boldsymbol{\xi}_{\beta'}\, E(h\boldsymbol{x}_\beta,\boldsymbol{\xi}_{\beta'})D_{\beta'}(\boldsymbol{\xi}_{\beta'},h\boldsymbol{x}') \\ &= D_0(\boldsymbol{x}_\beta,\boldsymbol{x}')-PV_{\boldsymbol{x}_\beta}\int_{Q_\beta}d\boldsymbol{\xi}_\beta\, E(\boldsymbol{x}_\beta,\boldsymbol{\xi}_\beta)D_{\beta'}(h\boldsymbol{\xi}_\beta,h\boldsymbol{x}') \end{aligned} \tag{4.117}$$

is the exact same form as (4.116), so $D_{\beta'}(h\boldsymbol{x},h\boldsymbol{x}')$ is another solution to this equation. By uniqueness,

$$D_{\beta'}(h\boldsymbol{x},h\boldsymbol{x}')=D_\beta(\boldsymbol{x},\boldsymbol{x}'). \tag{4.118}$$

This result on its own is rather interesting[†], as it means knowing the self-pressure on one object allows us to easily find the self-pressure on all congruent

[†]Though it is not very surprising. The self-pressure should only depend on the geometry of the object itself, which is the same for the two congruent objects Q_β and $Q_{\beta'}$.

objects. With this result, we have

$$\begin{aligned} V(h\boldsymbol{x}_\alpha, h\boldsymbol{x}_\beta) &= D_0(h\boldsymbol{x}_\alpha, h\boldsymbol{x}_\beta) - \int_{Q'_\beta} d\boldsymbol{\xi}_{\beta'}\, E(h\boldsymbol{x}_\alpha, \boldsymbol{\xi}_{\beta'}) D_{\beta'}(\boldsymbol{\xi}_{\beta'}, h\boldsymbol{x}') \\ &= D_0(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) - \int_{Q_\beta} d\boldsymbol{\xi}_\beta\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}_\beta) D_\beta(\boldsymbol{\xi}_\beta, \boldsymbol{x}') \\ &= V(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta). \end{aligned} \tag{4.119}$$

Finally it should be mentioned that we have assumed κ is constant. While we have not actually shown this in the thesis, we know that κ depends only on $\partial_n D_0$ (see (4.70)), so at least it will be invariant under h. Now we have

$$\begin{aligned} \kappa R(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) &= \kappa P(h\boldsymbol{x}_\alpha, h\boldsymbol{x}_\beta) \\ &= V(h\boldsymbol{x}_\alpha, h\boldsymbol{x}_\beta) - PV_{h\boldsymbol{x}_\alpha} \int_Q d\boldsymbol{\xi}'\, E(h\boldsymbol{x}_\alpha, \boldsymbol{\xi}') P(\boldsymbol{\xi}', h\boldsymbol{x}_\beta) \\ &= V(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) - PV_{\boldsymbol{x}_\alpha} \int_Q d\boldsymbol{\xi}\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}) R(\boldsymbol{\xi}, \boldsymbol{x}_\beta), \end{aligned} \tag{4.120}$$

which is what we wanted to show. Therefore,

$$P(h\boldsymbol{x}, \boldsymbol{x}') = P(\boldsymbol{x}, h^{-1}\boldsymbol{x}'). \tag{4.121}$$

As an example, consider a configuration of two concentric circles with radii r_1 and r_2. Parametrise these circles in terms of polar coordinates, designating points on Q_1 and Q_2 as $\boldsymbol{x}_1(\theta)$ and $\boldsymbol{x}_2(\theta)$ respectively for $\theta \in [0, 2\pi)$. Isometries are all the rotations around the centre, $h_\varphi \boldsymbol{x}_\gamma(\theta) = \boldsymbol{x}_\gamma(\theta + \varphi)$, as well as all reflections over any axis passing through the centre. Suppose we have found $P(\boldsymbol{x}_1(0), \boldsymbol{x}_\gamma(\theta'))$ for all $\theta' \in [0, 2\pi)$. Then we can easily find $P(\boldsymbol{x}_1(\theta), \boldsymbol{x}_\gamma(\theta'))$ for any θ and θ', according to

$$P(\boldsymbol{x}_1(\theta), \boldsymbol{x}_\gamma(\theta')) = P(h_\theta \boldsymbol{x}_1(0), \boldsymbol{x}_\gamma(\theta')) = P(\boldsymbol{x}_1(0), \boldsymbol{x}_\gamma(\theta' - \theta)), \tag{4.122}$$

whose value we already know. Similarly, knowing $P(\boldsymbol{x}_2(0), \boldsymbol{x}_\gamma(\theta'))$ enables us to easily find $P(\boldsymbol{x}_2(\theta), \boldsymbol{x}_\gamma(\theta'))$ for all θ.

More specifically, if H is the group of all isometries that preserve Q and S is a subset of Q that generates Q under the action of H, i.e. $H(S) = Q$, then it is sufficient to find $P(\boldsymbol{x}, \boldsymbol{x}')$ for $\boldsymbol{x} \in S$ and $\boldsymbol{x}' \in Q$. In the example of concentric circles above, one point on each circle is enough to generate the whole system under rotations.

There is another way to simplify further. Suppose $\boldsymbol{x}_\beta$ is a fixed point for $g \in H$, i.e. $g\boldsymbol{x}_\beta = \boldsymbol{x}_\beta$, for instance a reflection where $\boldsymbol{x}_\beta$ lies on the axis. Let $G \subset H$ be the subgroup containing all such isometries, and suppose that there is a subset $T \subset Q$ such that $G(T) = Q$. Then the integral in (4.109) can be reduced from an integral over Q to an integral over T, reducing the number of points in our discretization. As a more explicit example, suppose G is generated by $\{g_1, \ldots, g_n\}$ and that $T_j = g_j(T)$ such that $T_1 \cup \cdots \cup T_n = Q$ and $T_i \cap T_j = \emptyset$ for $i \neq j$. Then

$$\begin{aligned}\int_Q d\boldsymbol{\xi}\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}) P(\boldsymbol{\xi}, \boldsymbol{x}_\beta) &= \sum_j \int_{T_j} d\boldsymbol{\xi}_j\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}_j) P(\boldsymbol{\xi}_j, \boldsymbol{x}_\beta) \\ &= \sum_j \int_T d\boldsymbol{\xi}\, E(\boldsymbol{x}_\alpha, g_j\boldsymbol{\xi}) P(g_j\boldsymbol{\xi}, \boldsymbol{x}_\beta) \\ &= \int_T d\boldsymbol{\xi}\, E_G(\boldsymbol{x}_\alpha, \boldsymbol{\xi}) P(\boldsymbol{\xi}, \boldsymbol{x}_\beta),\end{aligned} \tag{4.123}$$

where

$$E_G(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_j E(\boldsymbol{x}, g_j\boldsymbol{\xi}). \tag{4.124}$$

This reduces the number of necessary discretization points when solving (4.109) by a factor of n. Although the number of evaluations of E increases by a factor n, the reduction in number of points improves the computation time by a factor n^2, leading to an overall net improvement by a factor n. This is especially relevant for three-dimensional cases; for example, for two concentric spheres, G is all the rotations around the axis through $\boldsymbol{x}_\beta$, and T becomes a meridian - essentially, this reduces the integration domain from a two-dimensional surface to a line segment.

4.8 The one-dimensional case

In this section we will study the case of zero-dimensional plates in one-dimensional space. We will consider the situation of two plates located at the points $x = 0$ and $x = a$, similar to what we did in Chapter 2. The purpose is to study how the method behaves in one dimension, which might give us insights such as verifying that the pressure integral is correct. In addition, it can be illustrative for understanding the method, and those who are unfamiliar

with the boundary integral method might want to read this section first before moving on to the more complicated treatment in Section 4.2 and onwards.

From (4.56) we have that the pressure on the plate at position x in one-dimensional space is

$$p(x) = \int_0^\infty \frac{d\omega}{2\pi} \omega^2 D(x, x, \omega). \tag{4.125}$$

The tangential derivative from (4.125) vanishes because there are no perpendicular dimensions. The Green's function $D(x, x')$ of the operator $L = \partial_{xx} - \omega^2$ satisfies

$$\partial_{xx} D - \omega^2 D = \delta(x - x'), \tag{4.126a}$$
$$\partial_x D(0, x') = \partial_x D(a, x') = 0 \tag{4.126b}$$
$$\partial_{x'} D(x, 0) = \partial_{x'} D(a, x') = 0. \tag{4.126c}$$

What we need to do is to find this Green's function, in particular we wish to find $D(0, 0)$ and $D(a, a)$, which is where our plates are located.

Introduce the free Green's function

$$D_0(x, x'') = -\frac{1}{2\omega} e^{-\omega|x-x''|} \tag{4.127}$$

satisfying $LD_0 = \delta(x - x'')$. We shall also need its derivative,

$$D_0'(x, x'') = \frac{1}{2} \operatorname{sign}(x - x'') e^{-\omega|x-x''|}. \tag{4.128}$$

For $x', x'' \in (0, a)$ we have

$$\begin{aligned} &\int_0^a dx \left(D(x, x') L D_0(x, x'') - D_0(x, x'') L D(x, x') \right) \\ &= \int_0^a dx \left(D(x, x') \delta(x - x'') - D_0(x, x'') \delta(x - x') \right) \\ &= D(x'', x') - D_0(x', x''). \end{aligned} \tag{4.129}$$

On the other hand, we also have

$$\begin{aligned}
&\int_0^a dx \Big[D(x,x')LD_0(x,x'') - D_0(x,x'')LD(x,x')\Big] \\
&= \int_0^a dx \frac{d}{dx}\Big[D(x,x')D_0'(x,x'') - D_0(x,x'')D'(x,x')\Big] \qquad (4.130)\\
&= \Big[D(x,x')D_0'(x,x'') - D_0(x,x'')D'(x,x')\Big]_{x=0}^{x=a} \\
&= D(a,x')D_0'(a,x'') - D(0,x')D_0'(0,x''),
\end{aligned}$$

where in the last step we have used the boundary condition

$$D'(0,x') = D'(a,x') = 0. \qquad (4.131)$$

Equating these two and rearranging gives[†]

$$D(x'',x') = D_0(x',x'') - D_0'(0,x'')D(0,x') + D_0'(a,x'')D(a,x'). \qquad (4.132)$$

We are interested in the limits as x' and x'' tend to the boundaries 0 and a. This gives four equations for the four variables $D(0,0)$, $D(a,0)$, $D(0,a)$ and $D(a,a)$. Note, however, that the pair variables $D(0,0)$ and $D(a,0)$ is decoupled from the other two; that is, if we for instance select $x' = 0$, it gives two equations, one for $x'' = 0$ and one for $x'' = a$, which we write on matrix form as

$$\begin{bmatrix} 1 + D_0'(0,0^+) & -D_0'(a,0) \\ D_0'(0,a) & 1 - D_0'(a,a^-) \end{bmatrix} \begin{bmatrix} D(0,0) \\ D(a,0) \end{bmatrix} = \begin{bmatrix} D_0(0,0) \\ D_0(0,a) \end{bmatrix}. \qquad (4.133)$$

If we select $x' = a$, it gives two similar equations for $D(0,a)$ and $D(a,a)$.

In taking the limits, pay special attention to D_0' as it is discontinuous when $x'' \to x$. To resolve this, we keep in mind that $x', x'' \in (0,a)$, and always use this to determine the direction of the limit. There is a term $D_0(x',x'')$ containing both x' and x'', but since D_0 is continuous, it does not matter which limit we take first. Thus there is no ambiguity in (4.133). Writing this

[†]Compare this to Section 4.2 - there, we got an integral equation, but the zero-dimensional case simplifies enormously as here we have a regular equation.

all out, we have

$$\begin{aligned} D_0(0,0) &= -\frac{1}{2\omega}, \\ D_0(0,a) &= -\frac{1}{2\omega}e^{-a\omega}, \\ D_0'(0,0^+) &= -D_0'(a,a^-) = -\frac{1}{2}, \\ D_0'(0,a) &= -D_0'(a,0) = -\frac{1}{2}e^{-a\omega}. \end{aligned} \tag{4.134}$$

Inserting this in (4.133), we get

$$\begin{bmatrix} 1 & -e^{-a\omega} \\ -e^{-a\omega} & 1 \end{bmatrix} \begin{bmatrix} D(0,0) \\ D(a,0) \end{bmatrix} = \begin{bmatrix} -\frac{1}{\omega} \\ -\frac{1}{\omega}e^{-a\omega} \end{bmatrix}, \tag{4.135}$$

which is straightforward to solve for $D(0,0)$, giving

$$D(0,0) = -\frac{1}{\omega}\frac{e^{2a\omega}+1}{e^{2a\omega}-1}. \tag{4.136}$$

However, inserting this in (4.125), we get

$$p(0) = -\frac{1}{2\pi}\int_0^\infty d\omega\, \omega \frac{e^{2a\omega}+1}{e^{2a\omega}-1}, \tag{4.137}$$

which does not converge! To regularize the integral, we will subtract the high frequency contribution.

In the limit $\omega \to \infty$, the exponential factors in (4.135) become insignificant, and the system becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} D_\infty(0,0) \\ D_\infty(a,0) \end{bmatrix} = \begin{bmatrix} -\frac{1}{\omega} \\ 0 \end{bmatrix}, \tag{4.138}$$

giving the high-frequency contribution

$$D_\infty(0,0) = -\frac{1}{\omega}. \tag{4.139}$$

Subtracting this from the pressure, the regularized pressure becomes

$$\begin{aligned} p(0) &= \frac{1}{2\pi}\int_0^\infty d\omega\, \omega^2 \left(D(0,0) - D_\infty(0,0)\right) \\ &= -\frac{1}{2\pi}\int_0^\infty d\omega \frac{2\omega}{e^{2a\omega}-1} = -\frac{\pi}{24a^2}, \end{aligned} \tag{4.140}$$

This matches the pressure in (2.148) found through mode expansion.

Note that this gives exactly the correct answer, although in Section 4.5 when we calculated the pressure on plates in two dimensions, we found that the numeric result differed from the correct solution by a factor of 1/2. This suggests that the source of the erroneous factor does not show up in the one-dimensional case. We elaborate on this point in Section 5.4.

Chapter 5

Results

5.1 BIM with von Neumann boundary conditions

In Chapter 4 we found an expression for the Casimir pressure, in terms of a Green's function given by a fully regularized boundary integral. The results have been tested numerically only for parallel plates, and there is indication that some piece might be missing for other configurations. In this section, we will state the result.

We are given a set of objects with surfaces $Q_1, \ldots, Q_r$. Let Q be the union of all surfaces, and let

$$D_0(\boldsymbol{x}, \boldsymbol{x}') = -\frac{1}{2\pi} K_0(\omega \|\boldsymbol{x} - \boldsymbol{x}'\|), \tag{5.1}$$

$$E(\boldsymbol{x}, \boldsymbol{x}') = -\frac{\omega}{2\pi} K_1(\omega \|\boldsymbol{x} - \boldsymbol{x}'\|)\, \boldsymbol{n}(\boldsymbol{x}') \cdot \frac{\boldsymbol{x} - \boldsymbol{x}'}{\|\boldsymbol{x} - \boldsymbol{x}'\|}. \tag{5.2}$$

where K_α is the modified Bessel function of second kind, of order α (see Appendix B). The Casimir pressure at a point is given as

$$p(\boldsymbol{x}) = \int_0^\infty \frac{d\omega}{2\pi} (\partial_{\boldsymbol{t}} \partial_{\boldsymbol{t}'} + \omega^2) P(\boldsymbol{x}, \boldsymbol{x}', \omega)\Big|_{\boldsymbol{x}'=\boldsymbol{x}} \tag{5.3}$$

The regularized Green's function $P(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta)$ on the boundary (with $\boldsymbol{x}_\alpha \in Q_\alpha$ and $\boldsymbol{x}_\beta \in Q_\beta$) is given by the boundary integral equation

$$\kappa P(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) = V(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) - PV_{\boldsymbol{x}_\alpha} \int_Q d\boldsymbol{\xi}_\gamma\, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}_\gamma) P(\boldsymbol{\xi}_\gamma, \boldsymbol{x}_\beta), \tag{5.4}$$

where the source V is

$$V(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) = D_0(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) - \int_{Q_\beta} d\boldsymbol{\xi}_\beta \, E(\boldsymbol{x}_\alpha, \boldsymbol{\xi}_\beta) D_\beta(\boldsymbol{\xi}_\beta, \boldsymbol{x}_\beta) \tag{5.5}$$

when $\alpha \neq \beta$ and zero when $\alpha = \beta$, and the self-pressure D_β is found by solving the boundary integral equation

$$D_\beta(\boldsymbol{x}_\beta, \boldsymbol{x}') = D_0(\boldsymbol{x}_\beta, \boldsymbol{x}') - PV_{\boldsymbol{x}_\beta} \int_{Q_\beta} d\boldsymbol{\xi}_\beta \, E(\boldsymbol{x}_\beta, \boldsymbol{\xi}_\beta) D_\beta(\boldsymbol{\xi}_\beta, \boldsymbol{x}'). \tag{5.6}$$

The term D_0 has a singularity at $\boldsymbol{x}' = \boldsymbol{x}_\beta$, but this singularity is integrable, so it will suffice to select $\boldsymbol{x}'$ close to $\boldsymbol{x}_\beta$.

We can discretize each object Q_α into N segments, $I_\alpha^1, \ldots, I_\alpha^N$. Let L_α^i be the length of I_α^i, and let $\boldsymbol{x}_\alpha^i$ be its midpoint. Let P be a block matrix with entries $P_{\alpha\beta}^{ij} = P(\boldsymbol{x}_\alpha^i, \boldsymbol{x}_\beta^j)$. Then P is given by the matrix equation

$$MP = V, \tag{5.7}$$

where M is a block matrix with entries

$$M_{\alpha\beta}^{ij} = \kappa \delta_{\alpha\beta} \delta_{ij} + (1 - \delta_{\alpha\beta}\delta_{ij}) L_\beta^j E(\boldsymbol{x}_\alpha^i, \boldsymbol{x}_\beta^j), \tag{5.8}$$

and V is a block vector with vector entries $V_{\alpha\beta}^{ij} = (1 - \delta_{\alpha\beta}) V(\boldsymbol{x}_\alpha^i, \boldsymbol{x}_\beta^j)$.

5.2 Failed numerical calculations

The most severe problem with our result is that it has not been shown to work in practice, except for the special case of parallel plates. An implementation for concentric circles was attempted, but the results did not match the predicted values from mode expansion.

Perhaps the most glaring issue is the coefficient κ. We were not able to determine this coefficient unambiguously, and we did in fact find that using the methods that were used in the Dirichlet case, we got the wrong result. It is possible that this coefficient is dependent on the curvature of the objects, which is why it chanced to give us the right answer for parallel plates. This point is discussed in much more detail in Section 5.3.

It is undeniably possible that the explanation is simply that the numerical implementation was incorrect. The procedure outlined in Section 5.1 is not

straightforward to implement as there are several details to pay attention to, such as the vectorizing procedure outlined in Section 4.4. In addition, running the code takes quite long time, making the process of testing it tedious, and it is difficult to test individual parts as we have no way of validating the intermediate results. It might be wise to design a source test to verify that the solution does indeed satisfy the Helmholtz equation. Such a source test was used by Kilen [14].

Another error source could be insufficient resolution N. Again, running time is an obstacle here, as a large resolution makes it very slow to compute our results. A reasonable approach would be to run the code with a low resolution to confirm that the result resembles the expected result at all, then use a supercomputer to work with higher resolution or more complicated geometries. It is also possible to consider an adaptive resolution: that is, to increase the resolution especially near the singular values of the functions, without increasing the resolution elsewhere.

5.3 The coefficient κ

In Section 4.2 we introduced the coefficient

$$\kappa = 1 + \lim_{\varepsilon \to 0} \int_{C_\varepsilon} d\boldsymbol{\xi}_\alpha \, \partial_n D_0(\boldsymbol{\xi}_\alpha, \boldsymbol{x}_\alpha), \tag{5.9}$$

where the integral is written out as

$$\begin{aligned} I &= \lim_{\varepsilon \to 0} \int_{C_\varepsilon} d\boldsymbol{\xi}_\alpha \, \partial_n D_0(\boldsymbol{\xi}_\alpha, \boldsymbol{x}_\alpha) \\ &= \lim_{\varepsilon \to 0} \frac{\omega}{2\pi} \int_{C_\varepsilon} d\boldsymbol{\xi}_\alpha \, K_1(\omega \|\boldsymbol{\xi} - \boldsymbol{x}_\alpha\|) \, \boldsymbol{n}(\boldsymbol{\xi}_\alpha) \cdot \frac{\boldsymbol{\xi}_\alpha - \boldsymbol{x}_\alpha}{\|\boldsymbol{\xi}_\alpha - \boldsymbol{x}_\alpha\|} \end{aligned} \tag{5.10}$$

and the integration path C_ε is a small interval around $\boldsymbol{x}_\alpha$ with radius ε. As $K_1(x)$ is singular at $x = 0$, this integral must be regularized, and how this should be done has not been made perfectly clear in this thesis.

In the Dirichlet case, a similar integral,

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon} d\boldsymbol{\xi}_\alpha \, D_0(\boldsymbol{\xi}_\alpha, \boldsymbol{x}_\alpha), \tag{5.11}$$

is encountered [14] [20]. In this case, the integral was regularized by modifying the path of the integral to exclude C_ε, and instead integrate along a

semicircular cavity with radius ε around $\boldsymbol{x}_\alpha$ (see Figure 5.1). This method was also used by Casimir and Polder in their original paper [5]. The integral was found to be of order $\mathcal{O}(\varepsilon \log \varepsilon)$ and tends to 0 as $\varepsilon \to 0$. We can try to apply the same technique: along the semicircle, let

$$\boldsymbol{d} = \boldsymbol{\xi}_\alpha - \boldsymbol{x}_\alpha = -\varepsilon \boldsymbol{n}(\boldsymbol{\xi}_\alpha). \tag{5.12}$$

Then

$$\frac{\omega}{2\pi} K_1(\omega \|\boldsymbol{d}\|)\, \boldsymbol{n}(\boldsymbol{\xi}_\alpha) \cdot \frac{\boldsymbol{d}}{\|\boldsymbol{d}\|} = -\frac{\omega}{2\pi} K_1(\omega\varepsilon). \tag{5.13}$$

Using the expansion $K_1(x) \approx \frac{1}{x} + \mathcal{O}(x)$, we can thus write the integral (5.10) as

$$\begin{aligned} I &= \lim_{\varepsilon \to 0} -\frac{\omega}{2\pi} K_1(\omega\varepsilon) \int_{C_\varepsilon} d\boldsymbol{\xi}_\alpha \\ &= \lim_{\varepsilon \to 0} -\frac{\omega}{2\pi} \left(\frac{1}{\omega\varepsilon} + \mathcal{O}(\varepsilon) \right) \cdot \pi\varepsilon = -\frac{1}{2}. \end{aligned} \tag{5.14}$$

Inserting this in (5.9) suggests that

$$\kappa = \frac{1}{2}. \tag{5.15}$$

This procedure was indeed performed at first in the numerical computations. However, when doing so, it was found that the results did not match the exact solution!

Let us look at a different regularization scheme. Taking a step back, what we really wanted to do in the first place was to evaluate the integral (see (4.64))

$$\lim_{\boldsymbol{x}'' \to \boldsymbol{x}_\alpha} \int_{Q_\alpha} d\boldsymbol{\xi}_\alpha\, D(\boldsymbol{\xi}_\alpha, \boldsymbol{x}') \partial_n D_0(\boldsymbol{\xi}_\alpha, \boldsymbol{x}''). \tag{5.16}$$

That is, we are interested in the limit as $\boldsymbol{x}'' \to \boldsymbol{x}_\alpha$, hence we can reinterpret (5.10) as

$$I = \lim_{\substack{\varepsilon \to 0 \\ \boldsymbol{x}'' \to \boldsymbol{x}_\alpha}} \int_{C_\varepsilon} d\boldsymbol{\xi}_\alpha\, \partial_n D_0(\boldsymbol{\xi}_\alpha, \boldsymbol{x}''), \tag{5.17}$$

and it becomes a matter of how to evaluate these limits. If we simply let $\varepsilon \to 0$ before $\boldsymbol{x}'' \to \boldsymbol{x}_\alpha$, we get $I = 0$ because the integrand is finite but the integration path becomes infinitesimal. But let us consider a less trivial case.

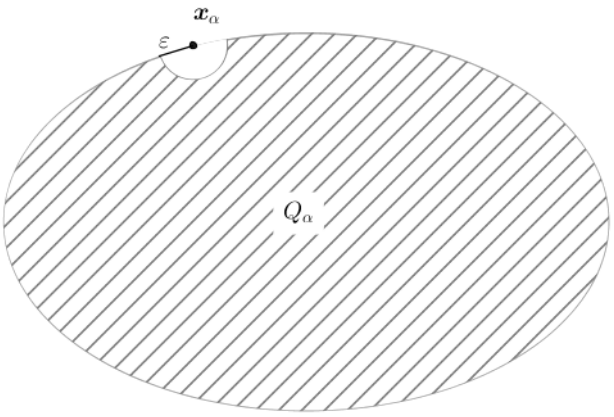


Figure 5.1: The object Q_α with a cavity around $\boldsymbol{x}_\alpha$ of radius ε

Let $\boldsymbol{x}'' = \boldsymbol{x}_\alpha + \eta(\varepsilon)\boldsymbol{n}_\alpha$, where $\eta(\varepsilon) > 0$ and $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Also, in the limit $\varepsilon \to 0$, we regard the curve C_ε as a straight line,

$$C_\varepsilon = \{\boldsymbol{x}_\alpha + s\boldsymbol{t}_\alpha \,|\, s \in [-\varepsilon, \varepsilon]\}\,, \tag{5.18}$$

where $\boldsymbol{t}_\alpha$ is the tangent unit vector at $\boldsymbol{x}_\alpha$. Then the line element is $d\boldsymbol{\xi}_\alpha = ds$ and

$$\begin{aligned} I &= \frac{\omega}{2\pi} \lim_{\substack{\varepsilon\to 0 \\ \boldsymbol{x}''\to\boldsymbol{x}_\alpha}} \int_{-\varepsilon}^{\varepsilon} ds\, K_1(\omega\|\eta\boldsymbol{n}_\alpha - s\boldsymbol{t}_\alpha\|)\, \boldsymbol{n}_\alpha \cdot \frac{\eta\boldsymbol{n}_\alpha - s\boldsymbol{t}_\alpha}{\|\eta\boldsymbol{n}_\alpha - s\boldsymbol{t}_\alpha\|} \\ &= \frac{\omega}{2\pi} \lim_{\substack{\varepsilon\to 0 \\ \boldsymbol{x}''\to\boldsymbol{x}_\alpha}} \int_{-\varepsilon}^{\varepsilon} ds\, K_1\left(\omega\sqrt{s^2+\eta^2}\right) \frac{\eta}{\sqrt{s^2+\eta^2}}. \end{aligned} \tag{5.19}$$

Again we use the Taylor expansion of K_1. All but the first term will vanish in the limit, and the integral can be evaluated exactly:

$$I = \frac{1}{2\pi} \lim_{\substack{\varepsilon\to 0 \\ \boldsymbol{x}''\to\boldsymbol{x}_\alpha}} \int_{-\varepsilon}^{\varepsilon} ds\, \frac{\eta}{s^2+\eta^2} = \frac{1}{\pi} \arctan\left(\frac{\varepsilon}{\eta}\right). \tag{5.20}$$

Depending on how we choose $\eta(\varepsilon)$, we can make the ratio ε/η converge to anything we like. E.g. for $\eta = \sqrt{\varepsilon}$ it converges to 0, for $\eta = k\varepsilon$ it converges to k, and for $\eta = \varepsilon^2$ it diverges to infinity. Since $\arctan(0) = 0$ and $\arctan(x) \to \pi/2$ as $x \to \infty$, all we can conclude from this calculation is that $I \in [0, 1/2]$.

The first method suggested $I = 1/2$, but in the implementation of the plates, it was found that $I = 0$ gave the correct answer. A very important question is whether I is the same for all configuration, or if it is actually geometry-dependent? It can be shown that $\partial_n D_0(\boldsymbol{x}_\alpha, \boldsymbol{x}')$ is proportional to the curvature at $\boldsymbol{x}_\alpha$ as $\boldsymbol{x}'$ comes close; could the value of κ also depend on the curvature? For straight plates with zero curvature, the integrand vanishes everywhere except exactly at $\boldsymbol{\xi}_\alpha = \boldsymbol{x}_\alpha$ where it is undefined. Introducing the semicircular cavity instead makes the integrand finite everywhere on C_ε. This point should be studied in more detail, for example by regularizing with a cavity as in Figure 5.1, then letting the curvature of the cavity change.

5.4 The missing factor 2

In Section 4.5 we calculated the Casimir pressure on parallel plates and found that the calculated value was one half of the predicted value. The exact same factor was found by both Kilen and Mikalsen [14] [20], in two and three dimensions with Dirichlet boundary conditions, and we have now also observed the same factor in the case of von Neumann boundary conditions.

Interestingly, in Section 4.8, we found an exact expression for the pressure in one dimension that matched the correct value perfectly. In this section, we started with the integral for the pressure (4.56), but the subsequent procedure was different from what we did in the two-dimensional case. This suggests either that the integral correctly gives the Casimir pressure and that something was lost in the regularization or discretization, or that the expression for the integral is in fact wrong, but that the error does not manifest in the one-dimensional case, for example if the expression was derived based on an assumption that is only true in one dimension.

In [14], Kilen used a source test to show that his process of finding D indeed correctly solved the Helmholtz equation. This indicates that the expression for the integral is wrong. If this is indeed the case, there could be an assumption that was made in its derivation that is valid only for one-dimensional space.

One particularity of how we treated the two-dimensional case is that we regularized by subtracting the self-pressure D_β. We defined D_β as the pressure when β was the only object. The physical interpretation is that objects exert an infinite pressure on themselves through self-interaction, but that only mutual interactions give rise to net forces. However, in the one-dimensional

case (and in several of the mode expansion regularizations), we subtracted the contribution from high frequencies. The high frequency waves will in reality penetrate the plates and give no contribution to the energy; to these waves, there essentially are *no* objects. This regularization is also based on the idea that there is an "infinite" amount of energy in the vacuum, and only the fluctuations give rise to forces. Therefore, since the force is related to energy through the gradient, subtracting a constant amount will not change the dynamics of the system. It would be interesting to look into whether it is possible to regularize the two-dimensional case by subtracting the high frequency contribution, both to see if this removes the factor 1/2, and more generally to see if this gives other useful results such as simplified formulae.

Finally it should be noted that if the factor of 1/2 is found to be the same for all dimensions, configurations and boundary conditions, then it is hardly a practical issue. Implementations of BIM can simply multiply the result by 2.

5.5 The problem with FIM

T. Emig claimed in [10] that for von Neumann Boundary conditions, the action is given by

$$\hat{S}[\varphi,\rho] = \int_{\mathbb{R}^d} dV(k^2|\varphi|^2 - |\nabla\varphi|^2) + \int_Q dA\,(\rho^*\partial_{\boldsymbol{n}}\varphi + \rho\partial_{\boldsymbol{n}}\varphi^*)\,, \tag{5.21}$$

which is the same result as we got in Section 3.3. Next it was claimed that the equations of motion for the classical field ϕ, satisfying $\delta\hat{S}_{cl}/\delta\phi = 0$, are given by

$$\nabla^2\phi + k^2\phi = 0, \quad \boldsymbol{x} \notin Q, \tag{5.22a}$$
$$\Delta\phi = -\rho, \qquad \boldsymbol{x} \in Q, \tag{5.22b}$$
$$\Delta\partial_{\boldsymbol{n}}\phi = 0, \qquad \boldsymbol{x} \in Q, \tag{5.22c}$$

where the Δ means

$$\Delta\phi = \phi_- - \phi_+, \tag{5.23}$$
$$\phi_\pm(\boldsymbol{x}) = \lim_{\varepsilon\to 0}\phi(\boldsymbol{x} \pm \varepsilon\boldsymbol{n}), \tag{5.24}$$

for $\boldsymbol{x} \in Q$, and that the corresponding classical action is

$$\hat{S}_{cl} = \frac{1}{2}\int_Q dA\,(\rho^*\partial_{\boldsymbol{n}}\phi + \rho\partial_{\boldsymbol{n}}\phi^*)\,. \tag{5.25}$$

We did however show in Section 3.4 that there is a major issue with this result: if the action is indeed (5.21), then $\delta\hat{S}_{cl}/\delta\phi$ implies that $\rho = 0$ everywhere, which is inconsistent with the fact that ρ is arbitrary. We also found that $\Delta\phi = +\rho$ in (5.22b), but this could be because we interpreted ρ differently from Emig.

The variation in the classical action can be written as

$$\delta\hat{S}_{cl}[\delta\phi^*] = \int_Q dA\,(\rho\,\partial_{\boldsymbol{n}}\delta\phi^* - \delta\phi^*\,\Delta\partial_{\boldsymbol{n}}\phi)\,. \tag{5.26}$$

This must be zero for all variations $\delta\phi^*$, by the principle of least action. However, it is possible that some variations satisfy $\delta\phi^* = 0$ on Q, but $\delta_{\boldsymbol{n}}\delta\phi^* \neq 0$. For example, if Q is a circle and we select $\delta\phi$ in polar coordinates as

$$\delta\phi(r,\theta) = \frac{R}{2\pi}\sin\left(\frac{2\pi r}{R}\right)\delta\vartheta(\theta), \tag{5.27}$$

where $\delta\vartheta(\theta)$ is a varation that only depends on θ, we get

$$\delta\hat{S}_{cl} = \int_Q dA\,\rho\,\delta\vartheta(\theta) = 0, \tag{5.28}$$

which suggests $\rho = 0$ everywhere, since $\delta\vartheta(\theta)$ is arbitrary. It is not clear where this problem comes from, but it seems it will inevitably arise from (5.21). One possibility is it is not possible to use the delta functional $\delta[\partial_n\varphi|_{\mathcal{Q}}]$ in this case. Again, it is not clear why this should not be possible, but the delta functional is not perfectly well understood.

In Section 3.4 we showed that if we disregard these problems and move on, we can derive a form similar to (5.22). The procedure relied heavily on the Dirac delta function and step functions, and a more rigorous treatment should employ the full theory of generalized functions.

In Section 3.7 we showed that by introducing the gradient field $\boldsymbol{v} = \nabla\varphi$, the boundary conditions become the Dirichlet conditions $\boldsymbol{n}\cdot\boldsymbol{v}|_{\mathcal{Q}} = 0$. This enabled us to avoid the issue in (5.26) and led us to the equations of motion for the classical field $\boldsymbol{u}$,

$$\nabla^2\boldsymbol{u} + k^2\boldsymbol{u} = -\nabla\times\boldsymbol{\pi}, \quad \boldsymbol{x}\notin Q, \tag{5.29a}$$

$$\Delta\boldsymbol{u} = 0, \qquad \boldsymbol{x}\in Q, \tag{5.29b}$$

$$\Delta\partial_{\boldsymbol{n}}\boldsymbol{u} = \boldsymbol{n}\rho - \boldsymbol{n}\times\Delta\boldsymbol{\pi}, \quad \boldsymbol{x}\in Q. \tag{5.29c}$$

The price to pay is that the equations of motion now included the inhomogeneous Helmholtz equation. This means that $\boldsymbol{v}$ expressed in terms of the Green's function G will contain an integral over the the volumes contained in the objects, thereby greatly increasing the computation cost. It might be worth mentioning that the field $\boldsymbol{\pi}$ comes from the condition that $\boldsymbol{v}$ must be a gradient field. If we lift this condition, the equations of motion become homogeneous, but it is not clear that they still describe the same physics.

Chapter 6

Conclusion

In this thesis, we have developed an expression for the Casimir pressure in terms of the Green's function D given by a fully regularized integral equation. We showed that applying this method to a configuration of parallel plates gives results that match the predicted values. However, it has not shown to be correct for other configurations.

Attempts were made at implementing the method for concentric circles, but it did not produce the right values for the pressure. Barring implementation errors, if the mathematical part is indeed wrong, the primary suspect error source is the coefficient κ. This coefficient can have different values depending on which regularization we choose, and it might be the case that it should not even be constant, e.g. it might depend on the curvature of the objects.

For parallel plates, the computed result was found to differ from the exact answer by a factor of $1/2$. In the Dirichlet case, it was also shown that a factor of 2 was missing, both for two and three dimensions. We see that this factor is also missing for the von Neumann case in two dimensions.

Further work

Before BIM for von Neumann boundary conditions can be called done, it must be shown that the procedure indeed gives results that are consistent with known values. When looking for error sources, the coefficient κ in particular should be put to scrutiny.

After developing this part of BIM, there are two natural ways to expand the method. The first is to apply it to von Neumann boundary conditions in three dimensions. Comparing the works of Mikalsen and Kilen, it is

clear that the 3D case is substantially more complicated than 2D due to stronger singularities that require additional regularization steps, and it is therefore reasonable to believe that extending the von Neumann case to three dimensions is also nontrivial. The second extension would be to consider even more general boundary conditions, such as Robin conditions.

Another interesting question is whether it makes a difference to regularize BIM by subtracting the high frequency contribution instead of subtracting the self-pressure. This alternative regularization could help illuminate the error sources, especially the missing factor 2, as well as providing insight in other aspects of the method.

Appendix A

The delta function and the delta functional

The famous Dirac delta function $\delta(x)$ is often defined as

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases} \tag{A.1}$$

This is not a perfectly rigorous definition. In fact, $\delta(x)$ is not even really a function, but rather what is called a *generalized function*. However, when working with the delta function, it is common to use heuristics. It can rigorously be defined as the generalized function with the properties

$$\delta(x) = 0 \text{ when } x \neq 0 \tag{A.2a}$$

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1. \tag{A.2b}$$

It can be shown that the generalized function with these properties is unique. These "axioms" can be used to show many interesting properties for the delta function, such as

$$\int_{-\infty}^{\infty} dx\, \delta(x-a) f(x) = f(a) \tag{A.3}$$

for any sufficiently well-behaved function f. A common way of writing the delta function is

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \, e^{i\lambda x}. \tag{A.4}$$

The delta function can be generalized in several ways. A very common generalization is to n variables, with the important property

$$\int_{\mathbb{R}^n} d\boldsymbol{x} \, \delta^n(\boldsymbol{x} - \boldsymbol{a}) \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{a}). \tag{A.5}$$

This delta function is often defined as

$$\delta^n(\boldsymbol{x} - \boldsymbol{a}) = \delta(x_1 - a_1)\delta(x_2 - a_2) \cdots \delta(x_n - a_n). \tag{A.6}$$

Another generalization is

$$\delta_S(\boldsymbol{x}) = \int_S d\boldsymbol{s} \, \delta^n(\boldsymbol{x} - \boldsymbol{s}), \tag{A.7}$$

where $S \subset \mathbb{R}^n$. The important property of this function is

$$\int_{\mathbb{R}^n} d\boldsymbol{x} \, \delta_S(\boldsymbol{x}) f(\boldsymbol{x}) = \int_S d\boldsymbol{s} \, f(\boldsymbol{s}), \tag{A.8}$$

which can be seen by exchanging the order of the integrals.

Intuitively, it might be helpful to think of a delta function as a condition. When integrating something multiplied by a delta function, we mean we are really interested in the points singled out by the delta functional. The definition of δ^n suggests that a product of delta functions means all the conditions must be satisfied. That is, the multiplication represents a logical conjunction of the conditions - an AND operation. Similarly, the definition of δ_S suggests that adding delta functions represents logical disjunction, an OR operation. This intuition will motivate the rest of our discussion.

In this section we will define the *delta functional*, a functional analogous to the delta function. Let φ be a function and let A be an operator on the functions. Given a boundary condition

$$A\varphi|_C = 0 \tag{A.9}$$

where C is a subset of the domain of φ, we define the delta functional $\delta[A\varphi|_C]$ with the property

$$\int D\varphi\, \delta[A\varphi|_C] F[\varphi] = \int D\psi\, F[\psi], \tag{A.10}$$

for any[†] functional F where the left-hand side is a functional integral over all functions, and the right-hand side is over functions ψ satisfying the boundary condition (A.9).

For a single point p_1, we define the delta functional as

$$\delta[A\varphi|_{\{p_1\}}] = \delta(A\varphi(p_1)). \tag{A.11}$$

where $\delta(x)$ is the usual Dirac delta function. For multiple points, the intersection of the conditions can be expressed as the product of the respective delta functions,

$$\delta[A\varphi|_{\{p_1,\ldots,p_n\}}] = \prod_{n=1}^{n} \delta(A\varphi(p_n)). \tag{A.12}$$

This is analogous to the definition of δ^n. We are interested in what happens when $n \to \infty$.

Assume C is compact. Let $\{C_n\}_{n=1}^{\infty}$ be a family of subsets of C, such that $|C_i| > |C_j|$ if $i > j$, and such that the distance between points in C_n become smaller as $n \to \infty$.[‡] The boundary condition $A\varphi|_C = 0$ can be approximated as

$$\lim_{n\to\infty} A\varphi|_{C_n} = 0, \tag{A.13}$$

We define the delta functional on a continuum as

$$\delta[A\varphi|_C] = \lim_{n\to\infty} \delta[A\varphi|_{C_n}] = \lim_{n\to\infty} \prod_{c\in C_n} \delta(A\varphi(c)). \tag{A.14}$$

The Dirac delta function can be expressed as

$$\delta(x) = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{i\lambda x}. \tag{A.15}$$

[†]We assume F is sufficiently well-behaved, but this notion is more intricate for functionals than real functions.

[‡]Formally, for any point $c^* \in C$ and $\varepsilon > 0$, there is an $M > 0$ such that for any $m \geq M$ there is a point $c_m \in C_m$ with $|c_m - c^*| < \varepsilon$.

Using this formula,

$$\begin{aligned}\delta[A\varphi|_C] &= \lim_{n\to\infty}\prod_{c\in C_n}\int_{-\infty}^{\infty}\frac{d\lambda_c}{2\pi}e^{i\lambda_c A\varphi(c)} \\ &= \lim_{n\to\infty}\int_{R^{|C_n|}}\left(\prod_{c\in C_n}\frac{d\lambda_c}{2\pi}\right)\exp\left[i\sum_{c\in C}\lambda_c A\varphi(c)\right]\end{aligned} \tag{A.16}$$

Let

$$\Delta x_n = \frac{|C|}{|C_n|} \tag{A.17}$$

where $|C|$ is the length or surface area of C (depending on the dimension), and let

$$\rho_c = \frac{\lambda_c}{\Delta x_n} \tag{A.18}$$

to get

$$\delta[A\varphi|_C] = \lim_{n\to\infty}\int\left(\prod_{c\in C_n}\frac{d\rho_c\Delta x_n}{2\pi}\right)\exp\left[i\sum_{c\in C_n}\Delta x_n\rho_c A\varphi(c)\right]. \tag{A.19}$$

As $n\to\infty$, we replace the product of integration variables by functional integration variables,

$$\lim_{n\to\infty}\prod_{c\in C_n}\frac{d\rho_c\Delta x_n}{2\pi} = D\rho\,. \tag{A.20}$$

The explanation of this process is not very well understood, and this is one of the theoretical weaknesses of the functional integral method. In the limit, the sum will be replaced by

$$\lim_{n\to\infty}\sum_{c\in C}\Delta x_n\rho_c A\varphi(c) = \int_C d\boldsymbol{x}\,\rho(\boldsymbol{x})A\varphi(\boldsymbol{x}). \tag{A.21}$$

Thus, the delta functional is

$$\delta[A\varphi|_C] = \int D\rho\, e^{i\int_C d\boldsymbol{x}\,\rho(\boldsymbol{x})A\varphi(\boldsymbol{x})}. \tag{A.22}$$

Two simultaneous conditions on a function,

$$A_1\varphi|_{C_1} = 0 \quad \text{and} \quad A_2\varphi|_{C_2}, \tag{A.23}$$

can be written as a product of Dirac delta functionals, since

$$\int D\varphi\, \delta[A_1\varphi|_{C_1}]\delta[A_2\varphi|_{C_2}]F[\varphi] = \int D\varphi_1\, \delta[A_2\varphi|_{C_2}]F[\varphi] = \int D\varphi_{1,2}\, F[\varphi], \tag{A.24}$$

where the subscripts j on $D\varphi_j$ indicate that integration should be over all functions satisfying the boundary condition $A_j\varphi|_{C_j} = 0$. The product of two delta functionals is

$$\delta[A_1\varphi|_{C_1}]\delta[A_2\psi|_{C_2}] = \int D\rho_1 D\rho_2\, e^{i\int_{C_1} d\boldsymbol{x}\, \rho_1(\boldsymbol{x})A_1\varphi(\boldsymbol{x}) + i\int_{C_2} d\boldsymbol{x}\, \rho_2(\boldsymbol{x})A_2\psi(\boldsymbol{x})}. \tag{A.25}$$

In particular, if the conditions are on the same curve,

$$\delta[A_1\varphi|_C]\delta[A_2\varphi|_C] = \int D\rho_1 D\rho_2\, e^{i\int_C d\boldsymbol{x}\, (\rho_1(\boldsymbol{x})A_1\varphi(\boldsymbol{x}) + \rho_2(\boldsymbol{x})A_2\varphi(\boldsymbol{x}))}. \tag{A.26}$$

A condition on a vector

$$A\boldsymbol{v}|_C = \mathbf{0} \tag{A.27}$$

can be viewed as simultaneous conditions on each of the components of $A\boldsymbol{v}$, and thus

$$\delta[A\boldsymbol{v}|_C] = \int D\boldsymbol{\rho}\, e^{i\int_C d\boldsymbol{x}\, \boldsymbol{\rho}(\boldsymbol{x})\cdot A\boldsymbol{v}(\boldsymbol{x})}. \tag{A.28}$$

Appendix B

Bessel functions

The Bessel functions are the canonical solutions to the differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0 \tag{B.1}$$

where α is a constant, generally any complex number, called the order of the equation. This equation is called Bessel's differential equation [1]. Since this is a second order differential equation, there are two linearly independent solutions.

B.1 The different types of Bessel functions

One solution to this equation is the Bessel function of the first kind $J_\alpha(x)$. This can be written as a series as

$$J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Pi(m+\alpha)} \left(\frac{x}{2}\right)^{2k+\alpha}, \tag{B.2}$$

where the pi function Π is the generalisation of the factorial. For positive non-integer α, J_α and $J_{-\alpha}$ are linearly independent, hence their span is the solution space of (B.1).

If the order is an integer n, the relationship

$$J_{-n}(x) = (-1)^n J_n(x) \tag{B.3}$$

holds, and the pair is no longer linearly independent. In this case, the Bessel functions of the second kind $Y_\alpha(x)$ give another linearly independent solution.

These are defined by

$$Y_\alpha(x) = \frac{J_\alpha(x)\cos\alpha\pi - J_{-\alpha}(x)}{\sin\alpha\pi} \tag{B.4}$$

for non-integer α, and as the limit of this expression for integer α.

Two other linearly independent solutions are the Hankel functions of the first and second kind,

$$\begin{aligned} H_\alpha^{(1)}(x) &= J_\alpha(x) + iY_\alpha(x), \\ H_\alpha^{(2)}(x) &= J_\alpha(x) - iY_\alpha(x). \end{aligned} \tag{B.5}$$

It might be illuminating to point out the analogy with the trigonometric functions. $\cos\alpha x$ and $\sin\alpha x$ are canonical solutions to $y''(x) + \alpha^2 y(x) = 0$. Another pair of solutions is $e^{i\alpha x}$ and $e^{-i\alpha x}$. The relationship between the two pairs is similar to the relationship between the Bessel and Hankel functions, and J_α and Y_α are related to each other similarly to how cos relates to sin.

It is also useful to have the modified Bessel functions of first and second type, denoted by I_α and K_α respectively. They are defined in terms of the aforementioned functions as

$$\begin{aligned} I_\alpha(x) &= i^{-\alpha}J_\alpha(ix), \\ K_\alpha(x) &= \frac{\pi}{2}\frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin\alpha\pi}. \end{aligned} \tag{B.6}$$

B.2 Relations between the functions

It can be shown that the modified Bessel function of second kind, K_α, can be written in terms of the Hankel functions:

$$\begin{aligned} K_\alpha(x) &= \frac{\pi}{2}i^{\alpha+1}H_\alpha^{(1)}(ix), && -\pi < \arg x \le \frac{\pi}{2}, \\ K_\alpha(x) &= \frac{\pi}{2}(-i)^{\alpha+1}H_\alpha^{(2)}(-ix), && \frac{\pi}{2} < \arg x \le \pi. \end{aligned} \tag{B.7}$$

With these equations, along with the definitions of I, K and H, we can solve for J and Y in terms of I and K:

$$\begin{aligned} J_\alpha(ix) &= i^\alpha I_\alpha(x) \\ Y_\alpha(ix) &= i^{\alpha+1}I_\alpha(x) - \frac{2}{\pi}i^\alpha K_\alpha(x). \end{aligned} \tag{B.8}$$

We have thus described how J and Y relates to I and K.

It is also interesting to see how the derivatives of the functions can be written in terms of the functions themselves. From the definition of J_α it can be shown that

$$J'_\alpha(x) = \frac{1}{2}\left(J_{\alpha-1}(x) - J_{\alpha+1}(x)\right). \tag{B.9}$$

Using this, it is straightforward to also show

$$\begin{aligned} Y'_\alpha(x) &= \frac{1}{2}\left(Y_{\alpha-1}(x) - Y_{\alpha+1}(x)\right) \\ I'_\alpha(x) &= \frac{1}{2}\left(I_{\alpha-1}(x) + I_{\alpha+1}(x)\right) \\ K'_\alpha(x) &= -\frac{1}{2}\left(K_{\alpha-1}(x) + K_{\alpha+1}(x)\right). \end{aligned} \tag{B.10}$$

In the special case $\alpha = 0$, using the relationship $J_{-1}(x) = -J_1(x)$ and similar relationships for the other functions, we can show that

$$\begin{aligned} J'_0(x) &= -J_1(x) \quad & Y'_0(x) &= -Y_1(x) \\ I'_0(x) &= +I_1(x) \quad & K'_0(x) &= -K_1(x). \end{aligned} \tag{B.11}$$

B.3 Asymptotic forms

For small x the functions J, Y, I, and K can be approximated as

$$\begin{aligned} J_0(x) &\approx 1 - \frac{1}{2}x^2 \quad & Y_0(x) &\approx \frac{2}{\pi}\left(\log\frac{x}{2} + \gamma\right) \\ J_\alpha(x) &\approx \frac{1}{\Pi(\alpha)}\left(\frac{x}{2}\right)^\alpha \quad & Y_\alpha(x) &\approx -\frac{\Pi(\alpha)}{\pi\alpha}\left(\frac{2}{x}\right)^\alpha \\ I_0(x) &\approx 1 + \frac{1}{2}x^2 \quad & K_0(x) &\approx -\log\frac{x}{2} - \gamma \\ I_\alpha(x) &\approx \frac{1}{\Pi(\alpha)}\left(\frac{x}{2}\right)^\alpha \quad & K_\alpha(x) &\approx \frac{\Pi(\alpha)}{2\alpha}\left(\frac{2}{x}\right)^\alpha, \end{aligned} \tag{B.12}$$

where $\gamma \approx 0.57721...$ is the Euler-Mascheroni constant and $\alpha > 0$. In the limit of large x, the functions J and Y oscillate while I and K have exponential behaviour. For $|\arg z| < \pi$,

$$\begin{aligned} J_\alpha(z) &= \sqrt{\frac{2}{z\pi}}\left[\cos\left(z - \frac{1}{4}\pi(1+2\alpha)\right) + e^{|\Im(z)|}\mathcal{O}(|z|^{-1})\right], \\ Y_\alpha(z) &= \sqrt{\frac{2}{z\pi}}\left[\sin\left(z - \frac{1}{4}\pi(1+2\alpha)\right) + e^{|\Im(z)|}\mathcal{O}(|z|^{-1})\right], \end{aligned} \tag{B.13}$$

and on the specified intervals,

$$\begin{aligned} I_\alpha(z) &= \frac{1}{\sqrt{2z}}\frac{e^z}{\sqrt{\pi}}\left(1+\mathcal{O}(z^{-1})\right), \qquad |\arg z| < \frac{\pi}{2}, \\ K_\alpha(z) &= \frac{1}{\sqrt{2z}}\frac{\sqrt{\pi}}{e^z}\left(1+\mathcal{O}(z^{-1})\right), \qquad |\arg z| < \frac{3\pi}{2}. \end{aligned} \tag{B.14}$$

The behaviours of the different functions are represented in Figure B.1.

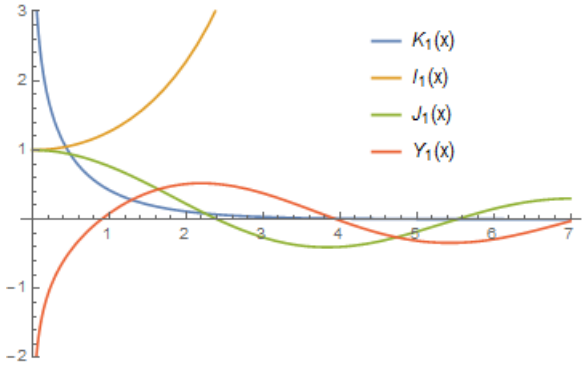


Figure B.1: The four Bessel functions for order $\alpha = 1$

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