Paper III

Anne Birgitte Fyhn: *A Climbing Class' Reinvention of Angles.* Accepted for publication by Educational Studies in Mathematics.

…so angles will be distinguished as different or equal before *measuring* angles is discussed. (Hans Freudenthal, 1983, p. 323)

A CLIMBING CLASS' REINVENTION OF ANGLES

Abstract

A previous study shows how a twelve-year-old girl discovers angles in her narrative from a climbing trip. Based on this research, the girl's class takes part in one day of climbing and half a day of follow-up work at school. The students' mathematise their climbing with respect to angles and they express themselves in texts and drawings. Their written and drawn expressions are categorised into three different levels: recognition, description and contextual tool. In addition, these expressions are interpreted to be narrative or analytical. All the narrative expressions were categorised as level one or below, while some of the analytical expressions were categorised as belonging to higher levels. The research findings point at how to use analytical drawings in work with analytical texts in geometry.

Key words

Analytical writing, angle, climbing, contextual tool, embodied cognition, flow, van Hiele, mathematising, reinvention

1. INTRODUCTION

When you are climbing, your body forms and reforms angles by making different shapes. The idea of the Climbing Class Project,' the CCP', is to let the students identify some of these angles and their consequences for the climbing. Based on how one twelve-year-old girl discovered angles in her climbing narrative, Fyhn (2006) claims that the climbing discourse can be a resource for the school geometry discourse.

The CCP introduces the girl's class to climbing as an integrated part of the teaching of angles. In the design of this experiment *"context problems are intended for supporting a reinvention process that enables students to come to grips with formal mathematics"* (Gravemeijer and Doorman, 1999). The goal of the CCP is to construct a route by which students can reinvent an operative angle concept. This paper focuses on the participating students during two days of work:

> *"How do students describe and explain angles in drawings and written text when they mathematise climbing with respect to angles?*

According to the PISA 2003 test (Kjærnsli et al., 2004), 'space and shape' is Norwegian students' weakest discipline within mathematics. They have a tendency to succeed better in reproduction than in more advanced cognitive competencies (ibid.). The Norwegian eighth graders' score in geometry at the TIMSS 2003 study were below the international mean. (Grønmo et al, 2004).

In Norwegian primary schools, angle teaching has until recently mainly been limited to measuring their sizes (Johnsen, 1996).

It is difficult for many Norwegian students to understand what an angle is, and how to compare angles of different sizes (Johnsen, 1996; Gjone and Norberg, 2001; Fyhn, 2004). In Gjone and Nordberg's (2001) study, only 30% of Norwegian sixth graders answered correctly which one is the largest and which one is the smallest of the angles in figure 1. Their study (ibid.) shows that among the other 70%, many students choose the same answer; that number 4 is largest and that number 1 is smallest.

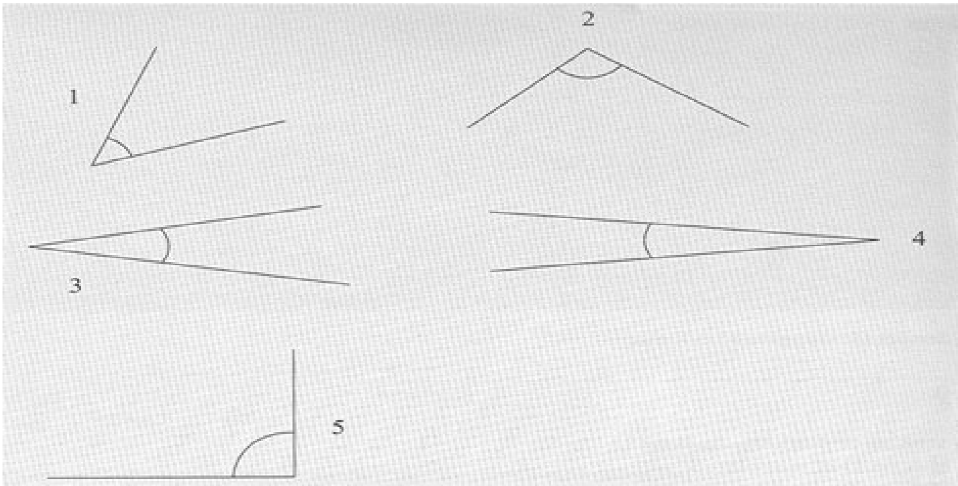

Figure 1. The angle task given in the pre-test. Questions: a) *Which one of the marked angles do you believe is the largest one?* b) *Which one of the marked angles do you believe is the smallest one?*

2. THEORETICAL FRAMEWORK

According to Lakoff and Núñez (2000, p. 365) *"Human mathematics is embodied, it is grounded in bodily experience in the world."* The term embodied can here be interpreted to mean either based on experiences from actions that involve the entire body, or based on experiences from using ones hands in manipulating objects. The CCP takes Freudenthal's (1991) perspective of mathematics as an activity, but the CCP's focus is delimited to require activities that are based upon experiences from use of the entire body.

Csikszentmihalyi (2000) describes 'flow' as the holistic sensation that people feel when they act with total involvement: *"The state of flow is felt when opportunities for action are in balance with the actor's skills"* (ibid., p. 49). Climbing provides a creative, enjoyable experience, and to a large degree one can choose in advance the level of challenge that best suits one's level of skills (ibid.).

The CCP intends to let students reinvent a concept of angle in a climbing context; angles as an integrated element of a (more or less) flow activity that involves the students' entire bodies.

2.1. Different Conceptions of Space

Berthelot and Salin (1998) divided the space into three main representations based on their sizes: microspace, which corresponds to grasping relations, mesospace, which corresponds to spatial experiences from everyday life situations, and macrospace, which corresponds to the mountains, the unknown city and rural spaces. When primary school students work with geometrical drawings on paper they use their microspace representation instead of some geometrical knowledge (ibid.). Climbing, however, takes place in mesospace.

Berthoz (2000) refers to "personal" space, "extrapersonal" space and "far" space where personal space, in principle, is located within the limits of a person's own body. According to Berthoz (ibid.), the brain uses two different frames of reference for representing the position of objects. The relationships between objects in a room can be encoded either as 'egocentric', by relating everything to yourself, or as 'allocentric', related to a frame of reference that is external to your body. Children first relate space to their own bodies and the ability of allocentric encoding appears later (ibid.).

When you are trying to ascend a passage of a climbing route, you encode the actual passage egocentrically within your personal space. But when you stand below a climbing route considering whether or how to ascend it, you exercise in allocentric encoding in extrapersonal space by considering how the route's different elements and your body relate to each other. Thus climbing can offer students good opportunities for moving back and forth between egocentric and allocentric representations.

2.2. Angles

Freudenthal (1983, p. 323) recommends introducing *"angle concepts in the plural because there are indeed several ones; various phenomenological approaches lead to various concepts though they may be closely connected."* He (ibid.) distinguishes between angle as a static pair of sides, as an enclosed planar or spatial part, and as the process of change of direction.

Lakoff and Núñez (2000) claim that angles existed in the early geometry paradigm where space was just the naturally continuous space in which we live our embodied lives. This supports work on the angle concepts in primary school as an integrated part of the students' physical activity.

According to Henderson and Taimina (2005, p. 38) *"it seems likely that no formal definition can capture all aspects of our experience of what an angle is."* They point out (ibid.) three different perspectives from which we can define angles: as a dynamic notion, as measure, and as a geometric shape. Angle as shape refers to what the angle looks like; angle as a visual gestalt.

Krainer (1993) uses four conceptions of angle:

a) angle without arc – angle as linked line (knee)

b) angle with arc (or angle space) – angle as part of a plane which is bounded by two rays

c) angle with arrow (or oriented angle space) – angle as part of a plane whose "creation" can be described by a rotation of a ray, and

d) angle with rotation arrow - angle describing the rotation of a ray (ibid., p. 79)

Mitchelmore and White (2000) found that the simplest angle concept was likely to be limited to situations where both the sides of the angle were visible; it is more difficult for children to identify angles in slopes, turns and other contexts where one or both sides of the angle are not visible.

The Norwegian word *angel* means fishing hook in English and according to Collins (2000) *angul* is an obsolete English word for fish-hook. The bent shapes of fishing hooks and corners can be recognized and described without use of any formal mathematics, but to measure an angle's size in degrees you need to be quite familiar with a formal way of measuring angles.

In the CCP, angles are shaped by three different elements in the climbing context:

a) the students' bodily joints

b) the ropes

c) the planes inside the building; the climbing walls, the floor and the roof

In addition, angles can be shaped between these different elements. Some of these angles are static while others are dynamic. The angles shaped by the bodily joints and by the ropes have neither arcs nor arrows. Moreover, neither of them have positive nor negative value. But angle as measure is a suitable category for analysing different ways of referring to angles in a climbing context. The CCP angles are divided into four categories:

i) angle as static shape

ii) angle as dynamic shape

iii) angle as measure

iv) angle as turn where one or two of the sides are invisible

A fifth category concerns the students' drawings: Angle with an arc. Angles from context c can only belong to categories i) and iii), while angles from contexts a and b can belong to other categories as well.

2.3. The Role of Context

"According to Freudenthal, mathematics can best be learned by doing … and mathematising is the core goal of mathematics education" (van den Heuvel-Panhuizen, 2003, p. 11). The term 'mathematising' is described as *"…the organising and structuring activity in which*

acquired knowledge and abilities are called upon in order to discover still unknown regularities, connections, structures" (Treffers, 1987, p. 247). The CCP intends to let the students mathematise angles in a climbing context.

> We distinguish horizontal and vertical mathematising in order to account for the difference between transforming a problem field into a mathematical problem on one hand, and processing within the mathematical system on the other hand (ibid.).

The CCP interprets whether the students' drawings show horizontal and/or vertical mathematising. A drawing from a climbing situation where arrows are drawn from the word 'angle' to a climbing person's bent joints will be interpreted as horizontal mathematising. In order to be interpreted as vertical mathematising a student's drawing must show either a stick-man, or an angle with an arc.

Freudenthal (1991) points out, that history tells us how mathematics was invented. The term guided reinvention means

> … striking a subtle balance between the freedom of inventing and the force of guiding, between allowing the learner to please himself and asking him to please the teacher. Moreover, the learner's free choice is already restricted by the "re" of "reinvention". The learner shall invent something that is new to him but well-known to the guide (ibid., p. 48).

The context plays a key role in the reinvention process (Gravemeijer and Doorman, 1999). One aim of the CCP reinvention process is to survey how students use angle as a tool for describing and explaining situations from a climbing context. Niss (1999, p 21) claims *"There is no automatic transfer from a solid knowledge of mathematical theory to… the ability to apply mathematics … in complex extra-mathematical contexts."*

2.4. Different Levels of Thinking

The CCP categorises the students' written texts and drawings as belonging to three different levels:

First level: the visual level. The student is able to recognise angles in a climbing context

Second level: the descriptive level. The student is able to describe the recognised angles

Third level: angle as contextual tool. The student is able to explain the described angles' logical consequences to their climbing.

The CCP denotes an operative angle concept as being able to make a written text, or a drawing that is categorised at the second level or above.

According to van Hiele (1986) we can discern between five different levels of thinking in mathematics:

First level: the visual level

Second level: the descriptive level

Third level: the theoretical level; with logical relations, geometry generated according to Euclid

Fourth level: formal logic, a study of the laws of logic

Fifth level: the nature of logical laws (ibid., p. 53)

The first CCP level is similar to the first van Hiele level, the student recognises angles. A statement that is interpreted to belong to this level is *"We have angles in our arms and legs, and the climbing wall is filled with angles".* A student who has reached the second level is able to describe angles, *"When I was up in the wall, I had a 90° angle in my foot";* the student here describes the angle shaped by his foot.

At the third CCP level students are able to use angles as tool for their logical reasoning about climbing; angles decide how hard it is to ascend a climbing route. An example of such reasoning is *"If you stand on your toes with your heel low, it is harder to raise one self than if you have the heel a little higher. The angle between the leg and the foot should be large."*

The second and third CCP levels are not quite similar to the second and third van Hiele levels; van Hiele claims that *"Discursive thinking, and thus explanation, for the most part uses the language of the second level"* (ibid., p. 86), while the third CCP level is constituted by context explanations. Pierre van Hiele's description of the levels concerns geometrical figures, while the CCP levels are designed as a tool for angles.

3. METHOD

The CCP is a case study where the students in one Norwegian seventh grade class participated in one day of climbing at the local indoor climbing wall and half a day doing follow-up activities in the classroom. Fyhn (2006) showed that a girl could be quite aware of *how* to climb a passage, without being able to explain *why* she moved as she did. The students in the girl's class constitute the informants in the CCP.

The two days were lead by the researcher who in her younger days was an eager climber. The class' mathematics teacher assisted, and a trainee teacher who was a skilled climber was responsible for the students' safety.

Nine girls and four boys in seventh grade constitute the CCP informants, while the entire class consisted of 18 students[1].

To investigate whether the students knew how to recognise the largest and smallest angles among others, they completed a pre-test and a post-test. The students' solutions to the pre-test task in figure 1 and a similar post-test task were analysed. These *"angle tasks"* dealt with identifying the smallest and the largest ones in a given group of angles. The pre-test was held about one week before the climbing while the post-test was held about one week afterwards.

The students were divided into three groups according to their success with the angle tasks in the pre-test and in the post-test:

- Group A: Succeeded in both of the tests
- Group B: Did not succeed in the pre-test, but succeeded in the post-test
- Group C: Did not succeed in neither of the tests

Table 1 shows how the thirteen students were categorised into the three groups. A widespread misconception is that a small angle has short sides and a large angle has long sides (Clements, 2003; Gjone and Norberg, 2001). Even though the sample of this study is very small, the regularity of the answers from the students in groups B and C indicates that the expected misconception to be present in the class.

Group and gender	N	pre-test – choice of answer		post-test- choice of answer	
		largest	smallest	largest	smallest
A girls	2	2*	4*	A*	B*
A boys	3	2*	4*	A*	B*
B girls	1	3 +4?	1+2?	A*	B*
B girls	1	4	1	A*	B*
B boys	1	5	1	A*	B*
C girls	4	4	1	A*	D
C girls	1	4	1	B	D

TABLE 1. The students' answers to the questions about the largest and the smallest angle in the pre-test and the post-test. The * sign marks the correct answers. The pre-test task is shown in figure 1. Angles B and D in the post-test corresponded respectively to angles 4 and 1 in the pre-test.

A week after the pre-test was completed the students spent one day at the local climbing wall, 'day one'. The climbing was top roping as shown in figure 2.

The next day at school, 'day two', three periods were spent doing follow-up work with angles; the students shaped different angles with their bodies, they looked at how the belay device functioned, and they drew a meso space version of the perpendicular bisection on the floor by use of a rope and a chalk.

The final activity on day two was to make a drawing from the climbing walls in their own way, and point at something concerning angles in the figure. As shown in table 2 eleven students made drawings this day.

Each of the two CCP mornings the students wrote about their expectations to that particular day, related to the day's two focus words, climbing and angles: 'What do you think this day is going to be like, related to climbing and angles?' At the end of each day they wrote about their experiences: 'What has the day been like, related to climbing and angles?'

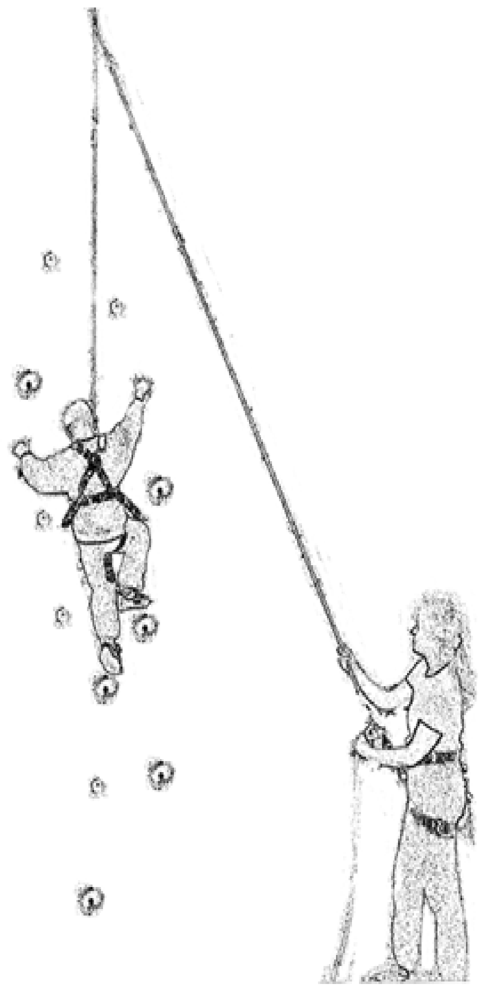

Figure 2. Top roping and belaying. The climber wears a harness which is attached to the end of a rope. The rope goes from this climber and through two carabiners that are fastened in the roof, and further down to one other person who is standing on the floor. The person on the floor is belaying the climber; she or he is continuously letting the rope pass through a belay device to keep it tight. That is to avoid the climber to reach the floor in case of falling. (Illustration by Ottar Fyhn Gohli and Knut O. Fyhn)

3.1. The Data and the Analyses

The CCP data is as follows:

- the students' solutions to the two tasks from the pre-test and the post-test
- each student's hand written texts about their expectations to and experiences from the two days.
- one boy's drawing from day one
- each student's drawings and writings from the last part of day two

The students' expectations were analysed with respect to whether they expected the days to be boring, and whether they expected to learn something that day. The students from groups A, B and C have their drawings and their writings from the end of each day analysed with respect to

- the four different angle categories i – iv
- the three levels: visual, descriptive and contextual tool

In addition the students' drawings are analysed with respect to

- horizontal and vertical mathematising
- the two categories narrative and analytical

The terms narrative and analytical are used according to Murphy and Elwood (1998), but are restricted to concern the students' drawings. A narrative drawing tells a (more or less continuous) story in a context; for example, an overview of a climbing situation. An analytical drawing, however, extracts angle(s) from context, and describes or explains these angle(s) as shown in figure 3. The categories narrative and analytical turned out to be inappropriate for analysing most of the students' writings.

Figure 3. Left: Part of David's drawing. Middle: Stick-man made by Laura. Right: How the rope is fastened to the roof, drawn by Laura. The Norwegian 'spiss vinkel' means 'acute angle'.

4. THE ANGLES AND THE CLIMBING

4.1. The Class' Work with Angles the Previous Year

The class' Norwegian teacher claimed that the class worked with angles in a practical way the previous year: *"The students cut out paper triangles, measured their angles and so on…"* This work is interpreted to indicate *"a dominant arithmetisation of geometrical conceptions. This arithmetisation contains the problem that interesting geometrical relations… are reduced to numbers and measures"* (Krainer, 1991, p. 259).

The previous year's work is interpreted to be based on a formal description of angles of different sizes; there was no sign of starting inductively in order to guide the students to find out what the word *'angle'* really meant.

4.2. The Students' Expectations

Most of the students had expected the climbing to be super or fun. This could be due to genuine interest in climbing, but it could be because they just enjoyed having visitors or being another place than at school. None of the students expected day one to be boring, and Sally in group C wrote: *"Maybe I will see maths from a different perspective"*.

Four students did not show positive expectations to day two; Eric, Olivia and Sally claimed statements that could be positive as well as negative. Carl in group A expected day two to be *"BORING"*. Maybe he was a clever boy who usually experienced the mathematics lessons as boring and with little challenge.

As opposed to the other students, those in group A did not expect to learn any mathematics from day one while both the girls in group A expected to learn mathematics from day two.

4.3. Students' Mathematising of Climbing - three Examples

Table 2 shows the students' drawn mathematising of climbing from the end of day two. The following three examples show how students mathematise climbing orally and by their bodies.

The room's two climbing walls, wall 1 and wall 2, meet in a corner. George from group A had ascended both an overhanging route in wall 1, 'route A', and a vertical route in wall 2, 'route B'. He was curious about whether one of these two routes ended up higher above the floor than the other one. He made a drawing of the two climbing walls and the corner where they meet.

George concluded with a statement: *"Route A has equal height as route B because the planks* (the horizontal panel in the slanting roof) *are parallel with the floor."* Here George is guided to claim a logic mathematical statement; he is giving an informal geometrical proof. The angles in his explanation are categorised into angle as static shape. He applies angles as a tool in a climbing context, and thus, his claim is interpreted into the third level. George's claim suits the third van Hiele level, too; *"... the pupil can deduce the equality of angles from* *the parallelism of lines"* (van Hiele, 1986, p. 42). His reasoning is based on at least one angle with only one visible side because the height of the overhanging wall has to be imagined.

George was here able to produce a mathematical statement about angles by himself, a statement that was interpreted to be at the third level. This in turn means that his angle concept is operative. Due to his reading and writing difficulties George had some special training and thus he was not present all of the time.

		A-analytical N-narrative	Level	Support by text	Mathematising: H-horizontal V-vertical	S- Stick-man A-Angle with arc
A	Tanya	N	2	text	H	
A	Maggie	N	2		H	
A	George	missing*				
A	Carl	A	2		H+V	A
A	Eric	missing				
B	Vicky	N/A	1	text	H+V	S
B	Olivia	N	0			
B	David	A	2		H+V	A
C	Sally	N	0	text		
C	Rita	N/A	1	text	H	
C	Peggy	A	2	text	H+V	S+A
C	Nelly	N	0			
C	Laura	A	2		H+V	S+A

TABLE 2. Analyses of the students' drawings. The rightmost column shows the vertical mathematising. 'missing' means that the student did not deliver any drawing.

* The drawing task was caused by George's drawing the first day

Rita in group C and two other girls entered the wall bars facing the wall. They were told to bend their elbows so their arms shaped acute angles. This task focused on angle as dynamic shape. Two of them immediately performed acute angles by their arms.

Rita almost made obtuse angles with her elbows as she watched her two classmates. Then she moved her feet a bit higher on the wall bars, but immediately, her hands were moved upwards, too. She looked at her arms and her elbows still formed obtuse angles. Then she moved her arms to a lower bar, but immediately her feet were placed on a lower bar. These elbow angles were categorised as angle as shape, just as Freudenthal (1983, p. 323) points out *"angles will be distinguished as different or equal before measuring angles is discussed."* This situation could be interpreted to show how this girl tried to copy the shape of her classmates' arms; that she had recognised the angles in their elbows and tried to bend her own elbows in a similar way.

This is an example of how the CCP's work with angles begins *"by offering the natural phenomena of spatial perceptions as the starting point of instruction"* (Treffers, 1987, p. 254). Pierre van Hiele (1986) claims about people approaching the first of his levels, *"They are guided by a visual network of relations; their intuition shows them the way"* (ibid., p. 50). Maybe Rita's knowledge about angles was quite all right; that she just did not feel comfortable being on the wall bar. However, Rita ascended the climbing wall, and that ascent required more motor skills than stepping up and down the wall bar. Maybe she just needed some time to make an egocentric representation of the allocentric representation that she observed.

Maggie was familiar with mathematising of climbing from before by taking part in the analyses of her climbing narrative (Fyhn, 2006). She explained: *"If you stand on your toe with your heel low, it is harder to raise yourself than if you have the heel a little higher. The angle between the leg and the foot should be large"* (ibid., p. 99). This oral logical statement is interpreted to be at the third level, and her angle is categorised into angle as dynamic shape. Maggie's *"proof"* here is informal; it is based on her climbing experiences.

4.4. The Students' Experiences

The students written experiences from each of the days are analysed separately. It turned out, in this small sample, that the most number of lines written by a boy was six while the shortest number of lines written by a group A girl was nine.

4.4.1. The Students in Group A

Eric hardly wrote nor drew anything at all throughout the entire project and his mathematics teacher explained this: *"He needs a lot of time to write; it is a work demanding patience for him to write anything at all."* Eric was the only one who did not climb. The CCP was lead by the researcher who had not met the students before, and thus, they were told that the climbing was voluntary.

On day one Carl just wrote *"When I was up in the wall I had a 90º angle in my foot."* He is interpreted to recognise the angle in his ankle and describe its size correctly; angle as measure at the second level. His sentence is interpreted to be what Murphy and Elwood (1998) denote as factual and analytical work.

George, who explained why the routes on wall 1 and wall 2 had equal heights, wrote *"the angles, I think it has turned out well."* His writing does not reflect the level of understanding that he showed in oral reasoning. A suggestion would be that some of his special lessons for reading and writing could focus on angles related to his climbing experience; through dialogues he could explain how and why he moved while he was climbing.

On day one the texts of the two girls in group A did not reflect any mathematics above the first level, but Maggie's writing included a good solution of the open task, *"There is an incredible amount of angles in everything if you just think about it!"*

Murphy and Elwood (1998, p. 174) point out *"Teachers seemed to reward and encourage narrative and descriptive writing over and above factual and analytical work."* Maybe Maggie was mainly used to narrative and descriptive writing; a "clever" school girl who did exactly what the teacher told her to do. Or maybe she was a skilled analytical writer who just preferred more chatty and narrative writing when she could choose genre.

On day two Tanya wrote: *"Angles are quite fascinating, and now I understand one hundred percent about angles in the climbing wall. We have angles in our arms and legs and the climbing wall is filled with angles."* Still Tanya's and Maggie's texts show no more mathematics than recognition of angles at the first level.

4.4.2. The Students in Group B

On day one David and Vicky just wrote that they had enjoyed working with angles but they did not claim to have recognised any angles. These writings are too general to be categorised as level 1. Olivia did not even mention the word angle in her four lines of text. David added a drawing of the perpendicular bisection which is categorised as level 1. According to Freudenthal (1991, p. 64) *"Name-giving is a first step towards consciousness."*

On day two both David and Vicky wrote something meaningful about angles while Olivia wrote twice that she had enjoyed working with angles. David's writing is interpreted to concern context c, and his last sentence indicates an approach towards the descriptive second level, *"Angles are important in building houses. Angles are used if you want something to become quite straight".* His text can be interpreted as an attempt to describe right angles' importance for building houses; an attempt to use angle as tool in a carpenter context.

Vicky wrote *"Angles are obviously everywhere for instance in the body, the blackboard, the door, the classroom, the nature +++"* which is interpreted to be recognition of angles; the first level, and it belongs to both context a, bodily joints and context c, the planes inside the building.

4.4.3. The Students in Group C

On day one they all wrote about angles except for one who did not deliver her report. But these writings were not interpreted to contain any mathematics; *"angles are fun"*, *"I enjoyed working with angles"* and so on. Added to her text Sally had drawn angles with arcs; 45°, 90° and 180°. Her angles are interpreted as angle as measure at the first level; she shows some knowledge about angles that is not related to the climbing context.

On day two the students wrote a lot more about angles than on day one. Peggy was the girl who got closest to the top of the overhanging wall, and she claimed that day two had been a bit boring and the progress had been a little too fast. This description fits into what Csikszentmihalyi (2000) denotes as a state of worry; the challenges exceed the skills. When the skills exceed the challenges the result is boredom. However, Norwegian students seem to use the word boring about both these states; that is quite reasonable because it is less humiliating to camouflage lack of skills as boredom.

Peggy had added two drawings to her text from day two; she drew a 90° angle and a stick-man doing the splits, under the split figure she wrote 180. These angles were without arcs.

4.5. The Students' Drawings

The students' drawings are microspace representations of the students' mesospace experiences. Carl's analytical drawing had a clear mathematical content and it presented angles in all three contexts; body, rope and walls. He presented 10 different examples of angles. His figures were interpreted to be at the second level; he described the angles of a rope that went through a belay device. Carl was the only one who referred to angles larger than 180°; on a bent elbow he had marked both the inner and the outer angle and his angles were with arcs. He was interpreted to have worked out this drawing systematically in order to cover as many different angles as possible, and most of them were shown in just one example.

The girls in group A both drew nice narrative overviews with lots of details as coloured clothes and detailed holds on the wall. Both these drawings showed angles as measure and the angles were without arcs. Tanya's drawing presented four different examples of angles while Maggie's drawing presented seven different examples. The girl's drawings showed angles in the body context and in the wall context; none of their angles included the rope context.

There are reasons to believe that both these girls were aware of angles in the rope context even though they did not draw it, maybe the school should have paid more attention to how to work systematically and analytically.

The analyses of these three group A students' drawings, indicate that the boy works analytically and focuses on fragments from the context while the girls work rather narratively. This can be interpreted to be what Murphy and Elwood (1998) explain, that girls less than boys, abstract issues from their context. This gender difference is hard for teachers to recognise (ibid.). It would probably not have been recognised in this case either if it had not been for the time spent analysing the students' drawings.

Vicky, in group B, drew overviews of four different climbing situations with arrows pointing at different angles, and the persons were all stick men. She added a text: *"This is an angle-land"*. Her angles are categorised as static shape and dynamic shape; angles from the contexts a and c. She does not refer to angles by their size in degrees. David drew three different fragments and one of them is presented in figure 3. Both Vicky's and David's writings from day two were interpreted as belonging to level 1 while Olivia who just wrote that angles were fun drew a nice coloured narrative overview of a climbing situation.

All of the group C students wrote something relevant about angles on their drawings. Both Peggy's and Laura's angles were interpreted to show all the three angle categories 'dynamic shape', 'static shape' and 'measure'. As shown in table 2, both Peggy's and Laura's analytical drawings are interpreted to express vertical mathematising opposed to the narrative drawings of the girls in group A.

Peggy sketched how the rope passed through the belay device and she marked one angle in that figure. The figure was supported by a text, *"Here is an angle from the rope when you act like this."* Furthermore the text was followed by a drawing that was difficult to interpret. Peggy also made a drawing of a climbing wall where the corners were marked with an arc and 90°. On the back of her sheet Peggy had written *"this was "insanely" exciting"*. This statement is interpreted to refer to day one because she reported that day two was boring.

5. DISCUSSION

5.1. Embodied Mathematics

Nemirovsky et al (2004) refers to 'manipulating materials with their hands and moving the materials around' as bodily activity. Watson and Tall (2002) refers to activities performed by their hands as "embodied action". The CCP interpretation of "embodied mathematics" requires that the students' entire bodies function as materials which they move around in meso space. The CCP offers a persisting alternation between what Berthoz (2000) denotes as *egocentric* representation in personal space and *allocentric* representation in extrapersonal space.

5.2. The CCP Related to the van Hiele Level Theory

An important focus of van Hiele's level theory is *"the problem of how to stimulate children to go from one level to the next"* (ibid., p. 5). Because the CCP interpretation of the term embodied was not basic for the van Hiele theories, the CCP chooses not to apply his learning stages.

One more problem with applying the van Hiele levels as angle categories lies in congruency; two equal angles are congruent whatever length of their sides. Van Hiele describes the third level thus *"a pupil having attained this level is able to apply congruence of geometric figures to prove certain properties of a total geometric configuration of which congruent figures are a part"* (ibid., p. 42).

5.3. The CCP Related to the Dutch RME

According to van den Heuvel-Panhuizen (2003) one of the basic concepts of the Dutch Realistic Mathematics Education, *'RME',* is Freudenthal's idea of mathematics as a human activity and that mathematising is the core goal of mathematics education. Another characteristic of RME is that *"Students pass through different levels of understanding on which mathematizing can take place"* (ibid., p. 12).

However, in RME the use of models are connected to the re-invention principle, while the CCP does not focus on models. The CCP has some demands that are not found in RME; the CCP focuses on students' *embodied* experiences from (more or less) *flow* activities in the *meso* space. In addition climbing talk is part of the CCP and that means alternating between *egocentric* representations in personal space and *allocentric* representations in extrapersonal space.

5.4. Limitations of the Climbing Approach to Angles

The point for the students is not to ascend to the top of a route but to challenge their own limitations. Indoor climbing walls can easily be changed and adapted to even the smallest and weakest child, only a small amount of students are excluded from taking part in climbing. In an ongoing study 58 of a total of 63 sixth grade students were present at a climbing day and all these 58 students participated in the climbing.

Students who cannot take part in climbing should be offered some other exciting activity like playing squash, or playing car racing games on a "play station", and then this activity could be mathematised with respect to angles in a similar way to the CCP.

The climbing approach to angles requires both a place to climb and some extra people like two competent parents, to take care of the students' safety during the climbing. Because of this extra need for resources, some teachers probably will probably choose another approach to the angle concept.

6. FINDINGS AND IMPLICATIONS

None of the climbers asked why they had to climb or what they needed the climbing experiences for. This could be what Csikszentmihalyi (2000) focuses on; enjoyment, here and now, and not as a reward for something in a dim and uncertain future.

The students who had the correct solution to the angle task in figure 1, consequently referred to angle as measure in their drawings and written texts. None of them delivered work that was categorized as level 3, but two of them claimed oral statements that were interpreted to be at the third level, and on those occasions their angles were not categorized as angle as measure. The class worked with measuring angles the previous year, thus the students' conceptions of angles before the start of the CCP were closely related to the angles' sizes in degrees.

6.1. Intended and Implemented Curricula

The group A students did not expect to learn any mathematics from the climbing day. Maybe they believed that practical activities were meant for students who struggled with mathematics; maybe they expected mathematics to take place in a classroom.

Work with geometry above the first recognition level could have been new to the class. This interpretation is supported by Clements' (2003) claim that the usual pre-school to middle school curriculum includes little more than recognising and naming geometrical shapes. This claim indicates that these curricula mainly concern geometry at level 1 and below.

The geometry part of the Norwegian curriculum of 1997 (KUF, 1996) focused to a great extent on properties of geometrical figures; geometry at level 2. The TIMSS 2003 (Grønmo et al, 2004) and PISA 2003 (Kjærnsli et al, 2004) results, as well as the CCP analyses, indicate that there is a long way to go before these intentions are implemented; explaining what goes on in a 'complex extra-mathematical context' (Niss, 1999) like climbing, is categorised as level 3.

6.2. Gender Differences

Two gender differences are found in this small sample of thirteen students. First: There are five girls and no boys in group C. Second: The girls in group A made narrative drawings at the end of day two, while no boys were interpreted to deliver a narrative drawing. The girls who made analytical drawings belonged to group C. As shown in table 2 five out of six analytical drawings are interpreted to show both horizontal and vertical mathematising at the second level. The narrative drawings either show no mathematising at all, or they show just horizontal mathematising.

6.3. Towards an Improved Design

Based on the CCP analyses, new and improved similar studies as well as larger scale studies can take place. Initially, an improved design will focus on the difference between narrative and analytical drawings, and thus students will need practise of making analytical drawings supported by texts. The angle-as-shape approach to angles is interpreted as useful in guiding students to the first level; at the end of the CCP only two of the participating students had not showed recognition of angles.

Before climbing the students can work in pairs and draw stick-men of each other in order to show bent bodily joints. The results in table 2 indicate that particularly the 'clever girls', the girls in group A, are expected to benefit from being aware of the difference between analytical and narrative drawing and from exercises in analytical drawing.

The DVD 'Angles in Climbing'[1] http://www.uvett.uit.no/temp/video/climbing.htm presents an introduction to an improved version of the CCP levels. The DVD should be introduced to teachers by someone who is familiar with the content of this paper.

In the improved version of the CCP levels, angles are related to the climbing context:

Level 1: The word 'angle' is related to recognition of angles

Level 2: Angles' shapes are described by the words 'acute', 'right' and 'obtuse'.

Level 3: Angles' sizes are explained by the words 'acute', 'right' and 'obtuse'. Statements about how some angle's size can decide how hard it is to ascend a climbing route.

Some months later the students had one more climbing day, and then one of the girls from group C wrote: *"The last time we were climbing I learned a lot, among other things, that when you climb you become more tired if your arms are held in a 90° angle than if they are stretched out."* This statement can be interpreted as belonging to the third level. However, this

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[1] [1] This URL wil be replaced by a new URL from Springer when the article is published

girl failed on the angle task in the post-test so maybe 90 degrees angle just was the name she used for angles at a certain shape. Thus one goal for the improved design is to include referring to angles' sizes as part of the level three language.

6.4. Future Possibilities

The CCP focuses on Freudenthal's (1973) assertion about geometry as grasping space, *"The space that the child must learn to know, explore, conquer, in order to live, breathe and move better in it"* (ibid., p. 403). The use of angles as a tool for analyses of climbing could be focused on by the research question: *"How can students analyse and improve their climbing technique by making analytical drawings and texts based on their own climbing experiences?"* For older students the work with this question could focus on vectors.

Norwegian students traditionally spend one week at a school camp during primary school and here climbing usually is a popular activity. Thus the above question could be focused on as an integrated part of students' school camp periods.

NOTES

[1] Three students who all succeeded well in the pre-test and who enjoy climbing as well were absent the first day and could not be part of the project. Two more students did not take part in the CCP as their parents did not sign the written permission. However, these two students joined their class both of the days.

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