A DISSERTATION FOR THE DEGREE OF PHILOSOPHIAE DOCTOR

Angles as Tool for Grasping Space:

Teaching of Angles Based on Students' Experiences with Physical Activities and Body Movement

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The term physical activity and body movement is used to have as broad a perspective as possible upon the students' use of their own bodies. When a climber stands still in the middle of a climbing route, situations can occur where the observer claims that quite a lot of physical activity goes on while there is no actual body movement. If a person stands on the ground and bends her or his arms, some observers will claim that no physical activity goes on; the person is just moving her or his arms.

To Marius, Sigurd, Yngvar and Ragnar

…geometry is grasping space. And since it is about the education of children, it is grasping that space in which the child lives, breathes and moves. The space that the child must learn to know, explore, conquer, in order to live, breathe and move better in it. (Freudenthal, 1973, p. 403)

If the students experience the process of reinventing mathematics as expanding common sense, then they will experience no dichotomy between everyday life experience and mathematics. Both will be part of the same reality (Gravemeijer and Doorman, 1999, p. 127).

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The most demanding part of the work with this thesis was the analyses in paper three. This work was carried out during the period from January to August 2006. I restarted climbing in the beginning of this period and I am convinced that my new climbing experiences influenced my research in a positive way; each time I managed to ascend an apparently impossible climbing passage I got the strength to carry on with the analyses of my data. Therefore I would like to thank my fellow climbers for just being there, joining me and sharing their experiences.

Tromsø, March 2007, Anne Birgitte Fyhn

List of papers and DVDs

Paper I

Fyhn, Anne Birgitte (2007).'Bridging outdoor Physical Activities with Written Work in Geometry'. Rewritten version of

Fyhn, Anne Birgitte (2004). 'How can experiences from physical activities in the snow influence geometry learning?' Paper at *ICME-10, Topic Study Group 10, Research and Development in the Teaching and Learning of Geometry.* DK: Copenhagen. Retrieved February 26, 2007 from http://www.icme-organisers.dk/tsg10/articulos/Fyhn_4_revised_paper.doc

Paper II

Fyhn, Anne Birgitte (2006). 'A climbing girl's reflections about angles', in *Journal of Mathematical Behavior* 25 (2006) 91-102

Paper III

Fyhn, Anne Birgitte (Accepted for publication). 'A Climbing Class' Reinvention of Angles'. Accepted for publication by *Educational Studies in Mathematics*.

Paper IV

Fyhn, Anne Birgitte (submitted manuscript). 'Climbing and Angles: A Study of how two Teachers Attain the Intentions of a Teaching Experiment'

DVD 1

Fyhn, Anne Birgitte (unpublished DVD). 'Geometry in the Snow and on Paper'

DVD 2

Fyhn, Anne Birgitte (Accepted for publication). 'Angles in Climbing'. Accepted as part of Paper III by *Educational Studies in Mathematics*.

Norwegian version:
Fyhn, Anne Birgitte. 'Vinkler i klatring'.

Table of Contents

1. Introduction

Maybe students who participate in the same leisure time activities succeed in the same domains of mathematics? Fyhn (2000) searched for and analysed relations between students' participation in different leisure time activities and their score in tasks in mathematics from TIMSS-1995 (Third International Mathematics and Science Study) and TIMSS-1998 (Repeat of the Third International Mathematics and Science Study). The categories of students were based on their regular participation at least once a week in different leisure time activities. Three categories were of particular interest for further research: *the creative-crafts-girls*, girls who participate in activities that concern drawing or handicraft; *the physically active students*, students who participate in at least one particular kind of physical activity; and *the skaters,* students who participate in snowboard or skateboard activities.

Geometry was expected to be a domain where the *creative-crafts-girls* had their highest score, but their score in geometry turned out to be rather low. These girls' highest scores were on tasks which tested the students' understanding of patterns. The *physically active students* had high scores on tasks that tested the students' understanding of rotation, tasks that indicate an understanding of space (ibid.). The *skaters*, too, had high scores on tasks that tested the students' understanding of rotation (ibid; Fyhn, 2005).

Despite the fact that the skaters' average marks in mathematics were significantly below the similar marks of the non-skaters, the skaters were interpreted as having achieved higher levels of understanding of rotation than the non-skaters. This was suggested to be a possible explanation as to why the skaters scored much higher than the non-skaters in a more difficult task that concerned rotation (ibid.).

1.1. Research Question

The above findings indicated that students' own experiences from physical activity and body movement could be a possible approach to the teaching of geometry. The project presented in this thesis is denoted as the Grasping Space Research Project 'the GSRP' and the research question is

How can the teaching of angles be based on the students' experiences with physical activity and body movement?

Most of the GSRP focuses on angles in a climbing context. However, the observant reader will probably find that similar approaches more or less can be carried out in other contexts as well; swimming, football, gymnastics, dance, skating and skiing, to mention a few.

The analyses of a pilot study (Fyhn, 2007) with focus on outdoor geometry with the students' use of compass, lead to a new choice of context (Fyhn, 2006) where the main research question was

> ***Is a climbing discourse a possible resource for a school-geometry discourse?***

Afterwards the students in one seventh grade class participated in a teaching experiment (Fyhn, accepted for publication a). In this study the research question was

> ***How do students describe and explain angles in drawings and written text when they mathematise climbing with respect to angles?***

The analyses of the participating teachers (Fyhn, submitted manuscript) focused on parts of the relevance and trustworthiness of the GSRP

> ***How do teachers attain students' mathematising of climbing as an approach to their teaching of angles?***

Because the GSRP intends to reach teachers and teacher educators with the research results two DVDs were made (Fyhn, unpublished DVD; accepted for publication b).

2. Theoretical Framework

2.1. What Is an Angle?

According to Henderson and Taimina (2005), an angle can be defined from three different perspectives: angle as geometric *shape*, as a *dynamic* notion (angle as movement) and angle as *measure*. The GSRP focuses on angle as turn and angle as shape in two different contexts.

2.1.1. The Word *Angle* – Angle as Shape

According to Freudenthal (1991, p. 64), "Space and the bodies around us are early mental objects… Name-giving is a first step towards consciousness." Thus some attention needs to be paid to the word angle.

The Oxford Dictionary (Fowler and Fowler, 1964) states that the word *angle* refers to the old French 'angle' or to the Latin word 'angulus' which is diminutive of the Greek 'agkos'; *bend*. The Norwegian word *angel* means 'fish-hook' in English and according to Walton (2003) *angling* is an art in which you use fish-hooks for catching fish. The Dutch word for *angle* is 'hoek' which is similar to *hook*.

A fish-hook has a bent shape and according to Henriksen and Haslerud (2002) there are several (Norwegian) words which refer to bent shape. These words are verbs as well as nouns, as shown in table 1.

nouns		verbs	
Norwegian	English	Norwegian	English
angel	fishhook	bøye	bend
bøy (1)	bend, curve	bøye (seg)	bend, stoop, bow (oneself)
bøy (2)	elbow, flexure	hekte	hook, fasten
hekte	hook	hekte	take in, pick up
hjørne (1)	corner	huke (1)	hook
hjørne (2)	mood, humour	huke (2)	catch, grab
huk (1)	corner, nook	huke (seg)	crouch, cover, bend (oneself)
huk (1)	crouch, squant	kroke	bend
knekk	bend, bow	krøke (1)	bend, crook
krok (1)	hook	krøke (2)	hook (catch a fish by accident)
krok (2)	corner, nook	stå i vinkel	be bent (1)
vinkel (1)	angle	ha knekk	be bent (2)
vinkel (2) (at a pipe)	elbow, knee	vinkle	angle

Table 1. Nouns and verbs referring to bent shape.

The Norwegian word for *angle* is 'vinkel' which in turn also means chevron in Norwegian. The German word 'Winkel' is similar to the Norwegian 'vinkel'. In Norwegian the *sides of an angle* are denoted as 'vinkelbein' (the angle's legs) while the German term is *Schenkel eines*

Winkels (the angle's thigh). The reflexive verbs listed in table 1 as well as the Norwegian and the German words for *sides of an angle* refer to something shaped by a human body.
The list of verbs as well as nouns in table 1, indicates that the word *angle* refers to both an action, to shape (something, by your own body or by your hands), and to *a* shape (which can be the result of an action). But according to a Norwegian Mathematics Dictionary (Thompson and Martinsson, 1997) *angle* is a noun and not a verb.

2.1.2. Ancient Angle Concepts

Freudenthal (1973) refers to Euclid's definition of angle:

> Euclid defines the angle as an inclination of lines…he meant halflines, because otherwise he would not be able to distinguish adjacent angles from each other… Euclid does not know zero angles, nor straight and bigger than straight angles…Euclid takes the liberty of adding angles beyond two and even four right angles; the result cannot be angles according to the original definitions…Nevertheless one feels that Euclid's angle concept is consistent (ibid., pp. 476-477).

Early humans used the stars and planets as they navigated over long distances, and ideas of trigonometry were developed by the Babylonians in their studies of heavenly bodies (Henderson and Taimina, 2005).

Trigonometry (2006) originates from the Greek words *trigonon* which means triangle and *metron* which means to measure. In trigonometry the angle is viewed as a centre angle of a circle where arcs and angles correspond to each other, "angles are measured by arcs, such that 360° and 2π correspond to each other" (Freudenthal, 1973, p. 477). This angle definition is more refined than Euclid's (ibid.).

2.1.3. Angle as Measure

In order to measure drawn angles and to draw angles at a given measure, the protractor is the classical instrument. However, the protractor is "an outrageously misleading instrument" (Freudenthal, 1983, p. 363) and it is "a trap the learner should have walked into once in order to avoid it in the future" (ibid., p. 363). This is explained as follows: "In order to measure angles, one has to subdivide angles. Subdividing angles can be confused with subdividing lengths or areas" (ibid., p. 363). Freudenthal warns against introducing angles by the measure approach: "angles will be distinguished as different or equal before measuring angles is discussed" (ibid., p. 323).

2.1.4. Angle as Turn

Freudenthal (1983) claims that the angle as turn "is the most natural, the most instinctive aspect of angle" (ibid., p. 328). The angle as turn is "the process of change of direction" (ibid., p. 327). However, the passing from the turn angle to the angle determined by two sides is abstract and difficult to attain and thus "teachers meet difficulties when they try to explain the turn angle" (ibid., p. 329). Freudenthal enlightens this problem by pointing out the case of turn angles where a full turn is counted as if it were nothing.

> ..two mental objects of angle, the relation between which is clear enough as long as one stays away from conceptualisation. In order to pass from the turn angle to the angle determined by two sides, one must abstract from … what happens in the meantime when one side is being turned into the other. Making this explicit can be difficult, indeed (ibid., p. 329).

The 'angle-as-turn' approach was carried out in the *Logo* software during the 1980s and 1990s. Several researchers examined how students explored the concepts of rotation and angle using *Logo* (Simmons and Cope, 1993). According to Simmons and Cope (ibid., p. 175) "it appears to be easier to use low-level procedural knowledge to generate action towards a solution than to think out a hypothesis which also requires the use of conceptual knowledge."

Simmons and Cope (ibid.) point out that future research needs to consider the nature of the instructional environment because by using *Logo* a large amount of students seemed to gain only instrumental understanding of both the programming concepts and the geometric concepts.

2.1.5. Angles in the Norwegian Curriculum

The curriculum of 1987 (KUD, 1987) introduced the angle as concept during the last three years of primary school and neither the protractor nor measuring angles were mentioned explicitly. But according to Johnsen (1996) the most frequently used way of working with angles in Norwegian schools was measurement and she claims that a large amount of Norwegian primary school students use the protractor in a wrong way. Her statement can be interpreted as a support to Freudenthal's (1983) claim regarding the protractor (2.1.3).

The curriculum of 1997 (KUF, 1996a; KUF, 1996b) is expected to have caused greater diversity in the Norwegian teachers' and students' work with angles because the measure approach was replaced by a dynamic approach. Henceforth a dynamic approach to angles was intended to be introduced in grade three, "Pupils should have the opportunity to …experience angles as rotating round a xed point, especially a quarter turn as a right angle" (KUF, 1996a, p. 161; KUF, 1996b).

In the curriculum of 2006 (KD, 2006a) the 'angles-as-measure' approach seems to have returned: The word *angle* occurs only once and that is under the subject area measurement for students at the fourth grade: "A goal for the teaching is that the student… is able to estimate and measure… angles" (ibid., p. 28, author's translation). This can be interpreted several ways: a) the angle-as-turn approach in the 1997 curriculum (KUF, 1996a; KUF, 1996b) was not a success among Norwegian teachers, b) this is a result of the government's strong demands for measuring the students' skills and c) research in the field didactics of mathematics is not treated with respect in Norway. According to Van den Heuvel-Panhuizen (2005, p. 13) "Measurement and geometry are two domains, each with their own nature." In the curriculum of 2006 (KD, 2006a) measurement and geometry occur as two different sections, but angles are only treated in the section measurement.

2.2. Embodied Geometry

Freudenthal (1991, pp. 75-76) claims

> No doubt once it was real progress when developers and teachers offered learners tangible material in order to teach them arithmetic of whole number… The best palpable material you can give the child is its own body.

Freudenthal here distinguishes between two kinds of teaching material, a) tangible material (manipulatives) and b) the students' own bodies. The tangible material often is so small that the students can place it on their desks and manipulate it with their hands.

According to Lakoff and Núñez (2000, p. 365) "Human mathematics is embodied, it is grounded in bodily experience in the world." The term embodied here can be interpreted to mean either based on experiences from actions that involve the entire body or based on experiences from using one's hands in manipulating objects.

2.2.1. Different Conceptions of Space

Berthelot and Salin (1998) divided the space into three main representations based on their sizes: microspace which corresponds to grasping relations, mesospace which corresponds to spatial experiences from daily life situations, and macrospace which corresponds to the mountains, the unknown city and rural spaces. When primary school students work with geometrical drawings on paper they use their microspace representation instead of some geometrical knowledge.

Berthoz (2000) refers to "personal" space, "extrapersonal" space and "far" space where personal space in principle is located within the limits of a person's own body.

According to Berthoz (ibid.) the brain uses two different frames of reference for representing the position of objects. The relationships between objects in a room can be encoded either ‘egocentric’, by relating everything to yourself, or an ‘allocentric’ way, related to a frame of reference that is external to your body. Only primates and humans are genuinely capable of allocentric encoding (ibid.).

> Moreover, allocentric encoding is constant with respect to a person’s own movement; thus it is well suited to internal mental simulation of displacements (ibid., p. 100).

Children first relate space to their own bodies and the ability of allocentric encoding appears later (ibid.).

When you are trying to ascend a passage of a climbing route you encode the actual passage egocentrically within your personal space. But when you stand below a climbing route considering whether or how to ascend it you exercise allocentric encoding in extrapersonal space by considering how the route’s different elements and your body relate to each other. Thus climbing can offer students good possibilities for moving back and forth between egocentric and allocentric representations.

2.2.2. An Interior Perspective and an Exterior Perspective

A blind student can experience how she or he shapes different angles by bending her or his elbows. A blind student can experience the difference between two hills of different steepness. Blind students can experience angle as turn as well as most of the angles listed in table 1 and many blind students can both draw angles and shape models of angles by the use of manipulatives. These are the actor’s experiences of angles from an interior perspective.

With the seeing student there is a danger that teachers forget this interior perspective. The seeing student can treat angles from an exterior perspective in two ways, a) from the observer’s point of view by watching and b) from the actor’s point of view through the use of manipulatives.

According to Nemirovsky, Borba and Dimattia (2004) the use of manipulatives (bodily activity) in mathematics education is part of a long tradition.

> However, there is an emerging perspective, sometimes called “Exploratory vision”, which describes vision as fully integrated with all the body senses and actions. Our eyes are constantly moving in irregular ways, momentarily fixing our gaze on a part of the environment and then jumping to another one. It is as if we are constantly posing questions to the visual environment and making bodily adjustments that might answer them (ibid., p. 304).

Watson and Tall (2002) refer to embodied action by a similar use of manipulatives; the students manipulate geometric figures by pushing them around on their desks. The term embodied here refers to the students' use of their hands.

A person's shadow is a visual mapping of this person's manipulation with her or his body. The actor can immediately observe the result of her or his manipulation. Gravemeijer (1997) here includes the visual image so that the action and the visual image are partly treated as a whole; the roles of the actor and the observer are interwoven.

According to de Moor (2005) solving simple geometry problems initially takes place through concrete experiences with eye and hand. Even though this indicates a shift between the roles of the actor and the observer, the students' physical experiences are restricted to the use of their hands.

2.2.3. The Students' Entire Bodies as Manipulatives

The use of tangible material (manipulatives) is denoted as bodily activity (Nemirovsky, Borba and Dimattia, 2004) and as embodied action (Watson and Tall, 2002). De Moor (2005) focuses on the development of spatial visualization and reasoning as a result of concrete experiences with eye and hand, but he does not label such experiences. However, neither of these is interpreted to treat students' entire bodies as manipulatives as Freudenthal (1991) suggested; they do not focus on the alternation between egocentric representations in personal space and allocentric representations in extrapersonal space (2.2.1).

Berthelot and Salin (1998) refer to mesospace actions where students move objects like benches and gymnasium mats. The manipulatives they use are too big to be placed on the students' desks and thus the students have to move their entire bodies in order to move the manipulatives. But still the manipulatives are objects outside themselves; their focus is not on how and why they move their bodies as they do.

Dewey (1998) points out, that mastery of the body is necessary for the child's development and that "such problems are both interesting and important, and solving them supplies a very genuine training of thinking power" (ibid., p. 206). However, Dewey is here interpreted to refer to the very small child as he refers to the joy the child shows in learning to use its limbs (ibid.).

The GSRP restricts the term embodied to the involvement of the students' entire bodies; the focus is on the teaching and learning of geometry, based on the students' experiences from alternating between egocentric representations in personal space and allocentric representations in extrapersonal space, moving their own entire bodies in mesospace. Physical activity and body movement requires a mastery of the relation with space that traditionally is focused in the Norwegian gymnastics lessons.

The GSRP intends to focus on how and why the students move their bodies in mesospace; how the use of their own bodies can lead to an alternation between egocentric representations in personal space and allocentric representations in extrapersonal space. The intention is to guide the students to make these experiences explicit.

2.3. Mathematics Education outside Norway

Norwegian students do not succeed particularly well in international tests in mathematics (Lie, Kjærnsli and Brekke, 1997; Mullis et al., 2000; Grønmo et al., 2004; Olsen et al., 2001; Kjærnsli et al., 2004).

If you coach a football team and your goal is to improve the team's results, then you study the international champions and their coaches; you look to elite teams from nations with a long football tradition like Brazil, England, Turkey, the Netherlands or Argentina. Or maybe you find it useful to study some of the interesting African teams like Ghana and Cameroon. But you definitely do not study your local neighbouring team just because someone you know claims that they have such a nice coach or that there is such a good atmosphere at their practices.

If you coach the mathematics education team of Norway and your goal is to improve the team's results, then you study the international champions and their coaches. Thus some attention will be paid to Japan, Singapore, Finland and the Netherlands because of their results from TIMSS (TIMSS 1995, TIMSS 1999 and TIMSS 2003: Trends in International Mathematics and Science Study) and PISA (Programme for International Student Assessment, 2000 and 2003).

This study will pay some attention to two Asian top score countries and two European top score countries from the TIMSS and PISA lists. For decades these four countries have systematically tried to improve their students' competence in mathematics and science and perhaps Norway has something to learn from them. These countries' rankings are shown in table 2. Actually, Finland and the Netherlands were the only non-Asian countries that scored higher than Japan in PISA 2003. Looking at the TIMSS 1999 and 2003 results, the three countries positioned between Singapore and Japan were all Asian.

The Netherlands turned out to be the most interesting of these four countries because their research tradition turned out to support the GSRP approach to teaching. The GSRP does not intend to create a copy of a Dutch school but to learn from the Dutch research and teaching tradition in order to improve what we do in Norway.

	Finland	Netherlands	Singapore	Japan	Norway
TIMSS 1995	-	-*	1	3	17
TIMSS 1999	14	7	1	5	-
TIMSS 2003	-	7	1	5	27
PISA 2000	4	-	-	1	17
PISA 2003	2	4	-	5	21

Table 2. The position of the five countries Finland, the Netherlands, Singapore, Japan and Norway regarding their scores in mathematics on the four most recent TIMSS and PISA tests. These TIMSS 1995 results refer to 13-year-olds, the TIMSS 1999 and 2003 results refer to 8th grade students while the PISA results refer to 15-year-olds. (Lie, Kjærnsli and Brekke, 1997; Mullis et al., 2000; Grønmo et al., 2004; Olsen et al., 2001; Kjærnsli et al., 2004).
* The Netherlands took part in the 1995 TIMSS test, but less than 75% of the selected schools participated.

2.3.1. Japan

Many Norwegian educationalists claim that it is a bad idea to study the Japanese education because so many Japanese youths commit suicide. In Norway this misconception about Japan seems difficult to challenge. The truth is

> The entire nation of Japan, with 126 million people, averages less than one murder among school-age children each year, plus a dozen or so of these suicides due to bullying. The United States, with two times as many people, has about five hundred times as many teenage murders (and roughly the same rate of teen suicide). (Reid, 2000, p. 131)

Despite the fact that in the Japanese mathematics lessons the students spend less time on routine work and more time on conceptual learning and reflection than students in Germany and the United States (Stiegler and Hiebert, 1999; Brekke, 2000), it still is difficult to find a Norwegian educationalist who thinks it is a good idea to study the Japanese school system.

When it comes to mathematics lessons the TIMSS 1999 video study (Hiebert et al, 2003) points out that Japan was the country with the highest frequency of presenting and examining alternative solution methods for mathematics problems.

The Japanese lesson study (Lewis, Perry and Murata, 2006; Stiegler and Hiebert, 1999) is a well established systematic way of improving the teaching of mathematics. According to Lewis, Perry and Murata (2006) lesson studies appeared at more than 335 U.S.

schools as a result of Stiegler and Hiebert's (1999) report from the video studies of TIMSS 1995.

2.3.2. Singapore

The Singaporean mathematics syllabus (MOE, 2006) focuses on problem solving and it pays much attention to metacognition. Problem solving differs from routine work and in addition the Singaporean syllabus (ibid.) focuses on students' attitudes towards mathematics. This indicates that there are elements in the Singaporean mathematics syllabus that can be interesting for Norway. Norwegian educationalists seem uninterested in what goes on in Singapore except for Sjøberg (2006) who claims that we neither may nor should copy the Singaporean school or society because Singaporean children live under very high pressure.

Norwegian football players do not claim that it is a bad idea to study Brazilian football even though many young Brazilian football players live under circumstances far below the Norwegian living standard. There is a great difference between studying the Singaporean syllabus and making attempts to copy their school and society.

2.3.3. Finland

Some Norwegian educationalists explained Finland's PISA scores by claiming that Finland had almost no immigrants at all. They ignored the fact that the PISA score in mathematics for Norwegian students who speak Norwegian at home was just the average OECD score (ibid.).

The domain problem solving in PISA has much in common with the domain reading (Kjærnsli et al, 2004). The Finnish students, who scored highest on the reading test in PISA 2000 (Olsen et al, 2000), scored significantly higher in the domain problem solving than in the domain mathematics on the PISA 2003 (Gille et al., 2004). And when Finland participated in TIMSS they did not enter the top ten list (Mullis et al., 2000). These results can be interpreted to mean that Finland's high mathematics score in PISA in some way is related to their reading ability and their high competence in metacognition.

The report from a committee back in 1987-1989 lead Finland to systematically aim to develop their mathematics curriculum (Kupari, 2004). In Norway, primary school teachers are also qualified for teaching mathematics in lower secondary school. But to be accepted as an applicant for the Finnish teacher education for lower secondary school you must have achieved one year full time university studies in mathematics.

Unlike Norway, Finland has a very small proportion of low achieving students in mathematics (ibid; Kjærnsli et al, 2004).

2.3.4. The Netherlands

Some Norwegian educationalists claim that the Netherlands' school system differs so much from the Norwegian system that these two countries are impossible to compare. The TIMSS 1999 video study (Hiebert et al., 2003) found that mathematics lessons in the eighth-grade in the Netherlands emphasised the relationships between mathematics and real-life situations to a greater extent than any of the other countries. The GSRP focuses on mathematics in real-life situations.

More of the Netherlands' research and teaching traditions will be presented in paragraph 2.4.

2.3.5. The Norwegian Attitude

The conclusion to this brief survey is that many Norwegian educationalists have negative attitudes towards learning from countries that have the highest scores on international tests in mathematics. Maybe these negative attitudes hinder Norwegian researchers' attempts to attain knowledge and information about how school children succeed in mathematics.

2.4. The Dutch Realistic Mathematics Education (RME)

The TIMSS and PISA results, presented in table 2, indicate that there could be good reasons for Norway to study the Dutch mathematics education. In addition, the Dutch language is closely related to Norwegian; to some extent many Norwegians intuitively are able to read Dutch texts. The Dutch answer to the worldwide felt need to reform the teaching of mathematics is denoted *RME* 'Realistic Mathematics Education' (Van den Heuvel-Panhuizen, 2003) and "In RME students should learn mathematics by developing and applying mathematical concepts and tools in daily-life problem situations that make sense to them" (ibid., p. 9)

In RME context problems are intended for supporting a reinvention process that enables students to come to grips with formal mathematics (Gravemeijer and Doorman, 1999). The RME takes Freudenthal's view of mathematics as an activity and not as a ready-made system (ibid.). A central characteristic of RME is what Freudenthal (1973; 1983; 1991) denoted as *mathematising;* "the "activity of organizing matter from reality or mathematical matter – which he called 'mathematization'" (Van den Heuvel-Panhuizen, 2003)

Mathematising and guided reinvention are central characteristics of both RME and the GSRP and will be focused later in this text (2.4.2; 2.4.3). In addition context plays an important role in the GSRP (2.5.8) as well as in RME. However, the GSRP uses a modified version of the Van Hiele levels where the RME focuses on models. One interpretation of this is that the RME use of models is developed from the Van Hiele level theory; the GSRP researcher's reinvention process of the Dutch RME is still under progress. Another interpretation is that the Van Hiele level theory is strong enough to survive.

2.4.1. The Roots: Freudenthal and the Van Hieles

The roots of the RME go back to Freudenthal and his colleagues in the 1970s (Van den Heuvel-Panhuizen, 2003). Freudenthal (1991, p. 96) claimed, "I owe the conception of the level structure of learning process to my collaboration with the Van Hieles, a couple who embodied, as it were, the marriage of theory and practice." And Pierre van Hiele, who worked at a Montessori secondary school from 1938 until 1951 claims, "In my Montessori period, it occurred to me to write a dissertation on a didactic subject" (Van Hiele, 1986, pp. 2-3). This indicates that in the 1970s the roots of the RME grew in fertile soil.

2.4.2. Mathematising

Freudenthal (1973) describes the term '*mathematising* something' as learning to organise this 'something' into a structure that is accessible to mathematical refinements:

> Grasping spatial *gestalts* as figures is mathematizing of space. Arranging the properties of a parallelogram such that a particular one pops up to base the others on it in order to arrive at a definition of parallelogram, that is mathematizing the conceptual field of the parallelogram (ibid., p. 133).

Treffers (1987) describes mathematising as an organising and structuring activity where acquired knowledge and abilities are called upon in order to discover still unknown regularities, connections and structures. Mathematising is an activity performed by the learners; the active learners discover regularities, connections and structures that still are unknown to them.

Treffers (ibid.) distinguishes between *horizontal* and *vertical* mathematising. "Horizontal mathematising leads from the world of life to the world of symbols" (Freudenthal, 1991, p. 41) while vertical mathematising is the more or less sophisticated process that goes on within the mathematical system; "Vertical mathematising is the most likely part of the learning process for the bonds with reality to be loosened and eventually cut" (ibid., p. 68). The distinction between horizontal and vertical mathematising depends on the involved person, the situation and the environments; "For the expert mathematician, mathematical objects can be part of his life in quite a different way but for the novice" (ibid., p. 42).

The RME takes Freudenthal's perspective of mathematics as a human activity, "According to Freudenthal, mathematics can best be learned by doing … and mathematising is the core goal of mathematics education" (Van den Heuvel-Panhuizen, 2003, p. 11). The level principle of RME is closely related to mathematising; students pass through different levels of understanding on which mathematising can take place. "Essential for this level theory of learning… is that the activity of mathematising on a lower level can be the subject of inquiry on a higher level" (ibid., p. 13).

2.4.3. From Doing to Thinking: Guided Reinvention

One more aspect of RME is *reinvention*; "that is recreating mathematical concepts and structures on the basis of intuitive notions…" (Treffers, 1987, p. 241). Freudenthal (1991) describes the term 'guided reinvention' as

> … striking a subtle balance between the freedom of inventing and the force of guiding, between allowing the learner to please himself and asking him to please the teacher. Moreover, the learner's free choice is already restricted by the "re" of "reinvention". The learner shall invent something that is new to him but well-known to the guide (ibid., p. 48).

As for reinventing geometry, Freudenthal (ibid.) claims that geometrical abstractions depend on contexts and he questions, "But how to link together nice pieces of geometrical reinvention, to get long chains of long-term learning processes, rather than leaving the learners with heaps of loose ends?" (ibid., p. 66).

The point of guiding the students through the reinvention process is to support them on their way from doing to thinking. A careful choice of context can prevent the learners from being left with what Freudenthal (ibid.) denotes as 'heaps of loose ends'; mathematising of students' body movement with respect to angles is an activity meant for students in primary school while mathematising of students' body movement with respect to vectors is an activity they can perform some years later in high school.

2.4.4. RME and Angles

Playing with light and shadows is one RME approach to geometry (Treffers, 1987; Van den Heuvel-Panhuizen et al., 2005) and this approach gives possibilities for mesospace work with angles as well as for an alternation between egocentric representations in personal space and allocentric representations in extrapersonal space. However, a person's shadow is a two dimensional mapping of her or his three dimensional body and thus such a mapping differs from the concrete body itself.

2.4.5. The GSRP's Intention: Students' Mathematising and Reinvention

The GSRP's original intentions were to a) search for a physical activity (context) that can be denoted as meaningful both for students in primary school as well as for those in upper secondary school and b) let students mathematise this context with respect to the part of geometry that concerns rotation. The results from paper one (Fyhn, 2007) lead to a new version b): to let students re-invent the conceptions of angle by mathematising the actual context with respect to angles.

Freudenthal (1991) claims that geometrical abstractions depend on contexts. One intention of the GSRP is to find a suitable context for introducing a concept that many students strive to understand; a suitable context here means a context that can lead to a vertical (Treffers, 1987) approach: "The principle of vertical planning is based on the notion that the 'lower' activity offers a necessary basis of experience for the 'higher' activity" (ibid., p. 62).

2.5. The Van Hiele Levels of Understanding

According to Van Hiele (1986) we can discern between five different levels of thinking in mathematics:

> First level: the visual level
> Second level: the descriptive level
> Third level: the theoretical level; with logical relations, geometry generated according to Euclid
> Fourth level: formal logic, a study of the laws of logic
> Fifth level: the nature of logical laws (ibid., p. 53)

On their way towards a certain level Van Hiele denotes the child to be in the corresponding period; a child who has not attained the first level is in the first period, a child who has not attained the second level is in the second period and so on (ibid.).

Pierre van Hiele (ibid.) claims about people approaching the first level, "They are guided by a visual network of relations; their intuition shows them the way" (ibid., p. 50). This can be illustrated by an example; if the naïve beginner in mathematics claims that a figure is a rhombus, ".. he probably does not mean any more than: 'This figure has the shape I have learned to call 'rhomb''" (ibid., p. 109).

According to Van Hiele (ibid.) shape is prominent at the second descriptive level. The second level is attained as the result of an analysis, "”This is not a rhombus, for the four sides are not equal"; "neither is this one, for this is a square"" (ibid., p. 50).

At the third level the students are able to perform informal deduction, "… the pupil can deduce the equality of angles from the parallelism of lines" (ibid., p. 42) and "definition is a concept of the vocabulary of the third level" (ibid., p. 84).

Van Hiele (ibid.) points out that the transition from one level to the next is dependent on instruction; it takes place under influence of a teaching-learning program. In the learning process leading to a higher level, there are five stages (ibid.)

1. *Information*. Students get acquainted with the working domain.
2. *Guided orientation*. Different relations of the network have to be formed.
3. *Explication.* Students try to explain relations in words.
4. *Free orientation*. Students learn to find their own ways in the network of relations.
5. *Integration*. Students build an overview of what they have learned (ibid.).

2.5.1. Vertical Planning

According to Van Hiele (1986) it was Piaget who first introduced levels, the experimental person at the lower level not understanding the leader at the higher level. But "The psychology of Piaget was one of development and not one of learning" (ibid., p. 5). Treffers (1987) points out that "the principle of vertical planning is based on the notion that the 'lower' activity offers a necessary basis of experience for the 'higher' activity" (ibid., p. 62). He further claims that even though the idea of vertical planning belongs to Bruner, it was Dienes who gave this idea meaning for mathematics education (ibid.).

Van Hiele claimed that true learning on one level is not possible as long as the learning process at the lower level has not sufficiently been completed (ibid.), "Van Hiele also knows this kind of phasing where acting on one level is subjected to reflecting and exploration in the next phase" (ibid., p. 275).

2.5.2. The Importance of Language

Language plays an important role in the Van Hiele (1986) level theory

> With the language that makes it possible to speak about the structures comes the possibility of describing the superstructures by reproducing the links between the given structures. After this, one can attain a higher level of thinking (ibid., p. 79)

According to Van Hiele (1986), Piaget did not see the very important role of language in moving from one level to the next. Dienes (1973) is also interpreted as not seeing the important role of language; Dienes' stages are organised so that language is not stressed before the fifth stage.

Even a dog can attain Dienes' (1973) stage three; abstraction. A dog can recognise similar shapes but a dog cannot describe these shapes; a dog cannot attain the second Van Hiele level. In the Van Hiele level theory, language plays an important role; each period includes a stage of 'explicitation' where the students "try to express them ('relations') in words, they learn the technical language accompanying the subject matter" (ibid., p. 54). The different use of language at the different levels is explained in a way that is familiar to many teachers:

> The learning process has stopped…The teacher does not succeed in explaining the teaching subject. He (and also the other pupils who have reached the new level) seems to speak a language which cannot be understood by the pupils who have not yet reached the new level. They might accept the explanations of the teacher, but the subject taught will not sink into their minds (Van Hiele, P.M. and Van Hiele-Geldof, D, 1958, p. 75).

2.5.3. The Role of Context

Van Hiele's levels were published in 1955 which is earlier than the roots of RME (2.4.1). Thus Van Hiele's use of context must be viewed in the light of the period of time. Van Hiele (1987) claims that in order to understand a new subject, its context must be totally clear:

> If we see geometry as science, we have no concern for space, nor for geometric figures in space, but only for the relations between properties of those figures…the context of a scientific study of geometry totally differs from the context at the introduction of the subject (ibid., p. 60).

The Van Hiele use of context is interpreted to differ from real life context; "the study of geometry… has very little to do with the way space is experienced" (ibid., p. 59). As for the role of 'real-life-context' both the GSRP and the RME are interpreted to differ from Van Hiele.

2.5.4. Inductive and Deductive Approaches

When the Van Hieles presented their level theories in 1955, they did not see the importance of the visual first level (Van Hiele, 1986). The reason for this could be what Van Hiele claims, "The transition from the base level to the second level is one from a level without a network of relations to a level that has such network" (ibid., p. 49).

The Van Hiele levels focus on the importance of the three successive levels of inductive work before the fourth formal deductive level is attained.

2.5.5. Dienes' Six Stages in the Learning Process

Through his analyses of the process of abstraction, Dienes (1973) concludes that there are six stages which need to be taken into account in the organization of mathematics education. The first stage concerns *free play*. Dienes' (1973) idea was to create an artificial environment to lead the child in more or less systematic ways to form logical concepts. The second stage concerns rules of the game and abstraction is the focus of the third stage which Dienes (ibid.) denotes as an *isomorphism game.*

To be able to talk about what she or he has abstracted the child needs a representation, and that is the focus of the fourth stage. Such a representation might be a stick-man, a construction by ruler and compasses, a Venn diagram, any other visual representations or even an auditory representation.

Stage five focuses on properties of the representations; it is needed to have a description of what is represented. To make a description a language is recommended:

> This is why the realization of the properties of the abstraction in this fifth stage must be accompanied by the invention of a language and then the use of this newly invented language to describe the representation (ibid., p. 8).

The sixth stage concerns 'theorems of the system'; this stage concerns mathematics beyond the primary school.

2.5.6. Van Hiele's Levels versus Dienes' Stages

Dienes' (1973) first stage concerns free play and here the notion of the environment seems to be of outstanding importance; "all learning is basically a process by which the organism adapts to its environment" (ibid., p. 6). The lower secondary school student does not distinguish between her or his 'environment' and her or his nearby part of what Lakoff and Núñez (2000) denote as the 'naturally continuous space'.

The GSRP focuses on students' mathematising of what Niss (1999) denotes as "complex extra-mathematical contexts". Dienes' contribution here is the free play stage which underlines the students' need for time to get familiar with the context.

The stages turned out to be less useful in analysing the students' mathematising for three reasons: a) the role of language, b) the use of artificial environment instead of a focus on context and c) Dienes did not require inductive work. Dienes' (1973) stages were then replaced by a framework which was based upon Van Hiele's (1986) levels of understanding (Fyhn, accepted for publication b).

2.5.7. Towards a framework for the GSRP

The fact that even a dog can reach Dienes' third stage points out the importance of non-verbal communication in mathematics education. While Van Hiele focuses on the role of language in students' development of their thinking, Dienes describes how far the student can reach without verbal reasoning. The GSRP focuses on language and thus Dienes' stages turn out to be discarded, in favour of the Van Hiele levels.

The Van Hiele levels could not be applied directly to the GSRP. The third Van Hiele level focuses on congruence while the third GSRP level is constituted by context explanations. Pierre van Hiele's description of the levels concerns geometrical figures while the GSRP levels are designed as a tool for angles. The three GSRP levels are

> First level: the visual level. The student is able to recognise angles in a climbing context.
> Second level: the descriptive level. The student is able to describe the recognised angles.
> Third level: angle as contextual tool. The student is able to explain why the described angles' sizes decide how hard it is to ascend a climbing route.

The GSRP meaning of an operative angle concept is to be able to make a written text or a drawing that is categorised at the second level or above.

2.5.8. The Climbing Context as an Inductive Approach to Angles

A modified version of the Van Hiele levels was chosen to function as a basis for a framework for analysing the students' learning of angles in a climbing context (Fyhn, accepted for publication a). The intention is to let students mathematise their own climbing with respect to angles and when the students mathematise a context they work inductively.

3. Methodology

The focus for the GSRP is the design of a teaching experiment, how the design works and how the analyses of it can lead to an improved design. Thus the GSRP is denoted as design research (Gravemeijer and Cobb, 2006; Cobb, 2001). However, the GSRP could have been denoted as action research (Krainer, 2006; Greenwood and Levin, 1998; Carr and Kemmis, 1986) or as lesson study (Lewis, Perry and Murata, 2006; Stiegler and Hiebert, 1999) as well as design research; actually these three kinds of research can hardly be treated as disjunctive because there are no clear and evident borders between them. Action research, lesson study and design research all follow self-reflective iterative cycles.

The GSRP initiative came from the researcher and there is no goal concerning teachers' development, nor did any teachers take part in the planning or in the analyses. Based on this the GSRP is not chosen to be treated as action research or as lesson study.

3.1. Design Research

Design research involves both instructional design and classroom-based research (Cobb, 2001). The goal is

> to develop sequences of instructional activities and associated tools, and to conduct analyses of the process of the students' learning and the means by which that learning is supported and organized (ibid., p. 456).

Design research requires no goal regarding the involved teachers' participation or development and the research focus is a teaching experiment designed by a research team (ibid.). According to Gravemeijer and Cobb (2006, p. 19) there are "three phases of conducting a design experiment, which are 1) preparing for the experiment, 2) experimenting in the classroom, and 3) conducting retrospective analyses."

3.1.1. Design Research as an Approach to the GSRP

Design research is chosen as the approach towards answering the research question because the main result of design research is the reasons how, why and to what extent the design works (Doorman, 2005). One goal of the GSRP is to design a local instruction theory that can function for teachers who are guiding students to reinvent an angle concept. The GSRP aims to be a theoretical as well as a practical contribution to the teaching and learning of angles (3.4.1.).

The background of the GSRP was a need for educational change combined with the researcher's curiosity regarding the outcomes of new approaches. In TIMSS 1995 the Norwegian average score in geometry among thirteen-year-olds was 47% while the international average was 51%. Based on this Lie, Kjærnsli and Brekke (1997) questioned the treatment of geometry in Norwegian schools.

The GSRP idea was to design research-based teaching experiment(s) that could be analysed and improved more or less continuously. Design research aims at creating innovative teaching (Gravemeijer and Cobb, 2006) and "the guiding question for the designer is: How could I have invented this? Here the designer will take into account his/her own knowledge and learning experience" (Gravemeijer and Doorman, 1999). The GSRP designer's knowledge and learning experiences regarding vectors as a tool for understanding and explaining climbing and outdoor life was essential for the design (Fyhn, 2006; Fyhn, 2000).

In design research the teachers usually perform the teaching while the research team makes "guidelines for the teacher" (Doorman, 2005, p. 76). In the GSRP the researcher gets closer to the intentions of the design because the researcher performs the teaching herself. This gives the researcher better possibilities of testing out how the teaching works, but on the other hand the researcher's possibilities of observation are limited.

Videotaping of the lessons could have provided better insight into what actually happened throughout the lessons. In further work based on the GSRP findings the improved design and its theoretical basis is carried out by teachers themselves (5.4.3).

3.1.2. Consequences of a 'Team' of One Single Researcher

The GSRP research 'team' consists of one single researcher while Gravemeijer and Cobb (2006) refer to research teams. Because of the lack of discussion partners the phases number one and three (3.1) probably were more difficult to carry out than if the researcher had cooperated in a team of college researchers.

A research team would probably have better possibilities of validating the data than one single researcher. On the other hand it is easier to keep an innovative focus if you do not have to explain it and defend it throughout good and bad periods of working with a project; in phases when new intuitive ideas are brought to surface they are touchy and vulnerable. Vulnerable innovative ideas need time to develop and thus they could risk being suppressed by some colleges' well-meaning initiatives or ideas.

To keep an innovative focus and at the same time keep an open-minded attitude towards college researchers is a question of balance. The result can be that more time is spent in dead-end streets but on the other hand the dead-end streets can provide insight.

According to Gravemeijer and Cobb (ibid.), the research group will take responsibility for the learning process of a group of students for a given period of time; five weeks, a whole school year, or something in between. When the GSRP researcher takes responsibility for the learning process of a group of students it is just for short periods of less than two weeks.

Gravemeijer and Cobb (ibid.) claim that the data sets typically include video-recordings of all classroom lessons and video-recorded interviews with all students before and after the lessons. The GSRP did not have the capacity to gather such a large amount of data and thus most of the data in papers three and four are limited respectively to the students' and the teachers' written and drawn material from a limited period of time. The data in the two single-case studies in papers one and two mainly consist of interviews.

3.2. Case Studies

One aim of the GSRP was to gather insight in to whether students' experiences with physical activity and body movement could function as a basis for the teaching of angles. In order to gather such insight a single case study was chosen as an approach (4.2; 4.4). A descriptive case study presents a complete description of a phenomenon in its context (Yin, 1993; Andersen, 2003). Clear descriptive case studies are almost non-existent (Andersen, 2003) and the single case studies in the GSRP were both exploratory and descriptive; there will always be some degree of interpretation even in the most objective description.

A multiple case study of the students in one class was chosen as a follow-up because a class is a suitable unit for teaching (4.5). The two participating teachers were quite different people with different backgrounds. A comparative case study was chosen to investigate how these two teachers responded to the actual approach to the teaching of angles (4.6). Andersen (2003) claims that regularities in the most different cases indicate robustness and possibilities of generalisation.

3.2.1. One Approach versus a Variety of Approaches

Freudenthal (1983) claims to "introduce angle concepts in the plural because there are indeed several ones; various phenomenological approaches lead to various concepts though they may be closely connected" (ibid., p. 323). Mitchelmore and White (2000) investigated how the structure of children's angle concept develops. They claimed that the fact that no textbook definition appears to match all physical angle contexts, emphasises the difficulty of forming a general standard angle concept.

Mitchelmore and White (2000) used nine situations from a variety of physical angle contexts. Four of these situations were movable while five were fixed. Mitchelmore and White (ibid.) conjecture that the students' standard angle concept is generalised during secondary school; "it would appear that the standard angle concept develops slowly" (ibid., p. 217). The GSRP does not focus on whether the angle concept develops slowly or not, but if necessary, a climbing day with focus on angles can be repeated as part of the welfare work in lower secondary school.

The GSRP started out with the 'angle as turn'-in-a-compass-context approach to the teaching and turned to the 'angle as shape'-in-a-climbing-context approach; the GSRP intends to let students work thoroughly with one single approach instead of getting a broad survey of angles in different contexts. The reason is similar to the reason for the choice of a single case study as a method for making the GSRP basis (Fyhn, 2007; 2006). Because the compass context was interpreted to be a failure (Fyhn, 2007), a new context was chosen.

> The need to use a case study arises when the empirical inquiry must examine a contemporary phenomenon in its real-life context, especially when the boundaries between phenomenon and context are not clearly evident (Yin, 1981, p.98).

In addition, the GSRP approach to angles is based on a) theories of teaching and learning of geometry and angles, b) the Norwegian Curriculum and c) the researcher's experiences of angles from different contexts.

The GSRP makes use of the plural angle concepts to underline the importance of work with angles in a variety of contexts. But the aim of this research is to give a thorough description of one approach to angles instead of giving a broad survey. The chosen approach to angles aims to suit a great proportion of the students. "Educators should also note the very great individual difference that exists; they should not try to force one pattern and model upon all" (Dewey, 1998, p. 228).

3.3. Relevance

As pointed out earlier in this text (2.3; 2.3.4) the results on international tests in mathematics indicated that Norway could benefit from studying the Dutch teaching and research tradition in the field of mathematics education. In addition, Norwegian schools have a need for educational change in geometry (3.1.1).

In November 2002, The Norwegian Ministry of Education and Research introduced a strategy plan for strengthening the subjects of mathematics, science and technology, *the strategy plan*. Once a year the cabinet minister releases a revised version of the strategy plan and the GSRP intends to pay attention to it.

3.3.1. Different Groups - Equal Goals?

The government and the politicians have their goals and their ideas about how to reach these goals. But researchers, teachers and students do not necessarily share these ideas. If the politicians ask for the researchers' advice they have to admit that they are not experts in the field. Because all politicians have been students themselves, many of them believe they are 'experts' in the field. When politicians choose to refer to research they often pick researchers arbitrarily, leaving some out.

Mellin-Olsen (1987) claims that whether a topic in a curriculum is regarded as important by a student depends on how the student relates it to her or his overall life situation. For the teacher this means to focus on the questions

> …how does the lesson relate to her pupils' conception of the important totalities of their world, and how can this lesson eventually transform this totality? (ibid., p. 33).

Based on this the GSRP will carefully consider the choice of context for the introduction of new topics.

3.3.2. GSRP and the Strategy Plan

The original strategy plan (UFD, 2002) pointed out some clear overall goals which remained unchanged in the second and third version of it. The GSRP aims to develop, analyse and improve new and research-based teaching of geometry and thus two of the strategy plan's goals were relevant for the GSRP:

- increased competence in mathematics, science and technology among students and teachers
- improved motivation among students and teachers towards including mathematics, science and technology as part of their education… (UFD, 2005, p. 9, author's translation).

According to the first version of the plan (UFD, 2002) school mathematics appears as a boring subject with a dull image. Unfortunately, neither the first strategy plan (UFD, 2002) nor the later versions of it (UFD, 2005; KD, 2006b) mention the Norwegian mathematics teachers' competence in the didactics of mathematics. In fact, Norway stands out internationally because a very small amount of mathematics teachers have a deeper formal understanding of mathematics or the didactics of mathematics (Grønmo, et al., 2004).

Investigating the second of the Norwegian Strategy plan's two listed overall goals (ibid.), one intention of the GSRP was to build the teaching upon some activity that the students performed just for fun. The GSRP intended to go beyond the positive attitudes and try to challenge the students' more cognitive beliefs because according to Goldin (2002) attitudes are just moderately stable predispositions that involve a balance of affect and cognition.

The government's latest strategy plan (KD, 2006b), however, has turned the second of the above overall goals into a more vague formulation "Instil positive attitudes to MST [mathematics, science and technology] among everyone in the educational system…" (ibid.,

p. 21). But still the GSRP can be interpreted to concern the intentions of the latest strategy plan.

3.3.3. Flow

Csikszentmihalyi (2000) introduced the concept *flow*, described as the holistic sensation that people feel when they act with total involvement. People seek flow for itself and not for any extrinsic rewards that might occur from it (ibid.):

> Achievement of a goal is important to mark one's performance but is not in itself satisfying. What keeps one going is the experience of acting outside the parameters of worry and boredom: the experience of flow (ibid. p. 38).

One quality of flow experiences is that they provide clear unambiguous feedback to the person's actions (ibid). Both mathematics and physical activities can be flow activities. However, there are certainly people who claim they experience flow from one of these subjects but not from the other one. What this study intends to do is to let students be introduced to geometry as part of their talk and reasoning about exciting leisure time activities that they enjoy taking part in. The ideal is to let the students experience that mathematics concerns flow activities.

'The students experiences with rotation of their own body' was a central element in the analyses and the findings of a master study (Fyhn, 2000), and thus the idea of this further research was to develop teaching of geometry based on the students' experiences from rotation of their own bodies, experiences that take place in a context which the participating students experience as exciting or fun.

3.4. Validity

Dorman (2005) describes four criteria that can validate the scientific basis for design research

> 1. The formulation and verification of testable conjectures about the students' development with respect to the educational environment created.
>
> 2. The *theoretical foundation* of the interpretative framework guides reduction and interpretations of the data.
>
> 3. The *credibility* of the instructional design, the data interpretation and the researcher's arguments.
>
> 4. The *engagement* of teachers and students during the teaching experiments (ibid.).

The conjectures about the students' development were included in the research questions in three of the papers, while in the fourth paper the research question concerns the teachers' attainment of the GSRP intentions (4.1).

The theoretical foundation of the interpretative framework guides the interpretations of the data in a way that results in iterative cycles of improved design; paper two is a result of the analyses of paper one, while papers three and four are results of the analyses of paper two. DVD 1 presents an improved design, the follow-up tasks are based upon the analyses of paper one. DVD 2 presents results of the analyses of paper three; both theories are about the process of the learning of angles and the means that are designed to support that learning.

The fact that paper two is published supports the credibility of the GSRP because both paper three and paper four are founded upon the results from this paper. Because the findings of paper one pointed towards a completely new approach, the main research question of paper two concerned the GSRP's credibility, "Is a climbing discourse a possible resource for a school- geometry discourse?" (Fyhn, 2006, p. 91).

Because the GSRP intends to be based upon students' experiences from flow activities (3.3.3), most of the participating students were engaged. That was so for the teachers as well; one of them was a gymnastics teacher and the other one was a professional climbing trainer and thus both of them enjoyed taking part in physical activities. However, the participants' engagement must be treated with caution (3.5.1).

3.4.1. The Replicability of the Papers and the DVDs

Papers one, three and four are hardly replicable because they involve teachers and teaching. "Teachers are professionals who continuously adjust their plans on the basis of ongoing assessments of their students' mathematical understanding" (Gravemeijer and Cobb, 2006, p. 44). In addition there are hardly two equal groups of students who are given equal conditions.

The second paper has quite good possibilities for replication but this paper does not focus on teaching; the paper is based upon the analyses of one single student's story. To a large extent a similar study can be carried out, but there is no guarantee that another climbing story will equal this one. There will be a need for finding an exposed climbing route that includes both a chimney, a dihedral and a passage that goes slanting upwards and there will be a need for nice weather and a student who succeeds well in mathematics.

The idea of DVD 1 is to create an instruction DVD in a way that makes it easy for teachers to replicate it with their own students. The DVD can function as a frame of reference for teachers who want to work inductively with mesospace embodied activities as an approach to geometry (2.2; 5.4.4). Teachers are also given written follow-up tasks that correspond with the curriculum's (KD, 2006a) demand for oral and written work in mathematics.

3.4.2. The Development of Theory

The intended result of design research is theory (Gravemeijer and Cobb, 2006):

> Design research aims for ecological validity, that is to say, (the description of) the results should provide a basis for adaptation to other situations. The premise is that an empirically grounded theory of how the intervention works accommodates this requirement… The intent is to develop a local instruction theory that can function as a frame of reference for teachers who want to adapt the corresponding instructional sequence to their own classrooms, and their personal objectives (ibid., pp. 44-45).

The idea of DVD 2 is to introduce teachers to a level theory of students' learning, because frameworks like the Van Hiele levels quite often are seen as belonging to the domain of researchers (Pegg, Gutiérrez, and Huerta, 1998).The DVD intends to be a local instruction theory that can function as a frame of reference for teachers who want to adapt the corresponding instructional sequence to their own classrooms (5.4.3). This DVD needs to be supported by either the corresponding text (Fyhn, accepted for publication a) or by an introduction by someone who has read it.

3.5. Ethical Aspects

3.5.1. The Participating Students

One important ethical aspect of doing research upon teachers and students is the use of time; to do classroom research you need to use some of the teachers' and students' time and you have no guarantee that their time is used in a way they feel comfortable with.

The qualitative approach in the GSRP intends to give the participants more or less direct response, but the final analyses have to be carried out before the result can reach the participants. This means that the students' experiences of participating in the research are similar to experiences from participating in developmental work. There is no reason to believe that different approaches to the teaching of angles will be of any interest to the participating students.

The research cannot be characterised as a waste of time for the students if they declare they enjoy taking part in the project. However, in carrying out flow activities as part of the teaching, there is an extra need for control of students' learning; the moral aspect of occupying the students' mathematics lessons must not get out of the researcher's focus.

The trouble is if students experience having fun and no mathematics while the teacher claims that they are having fun while they are learning mathematics. Then the worst case scenario can occur: the students' self confidence in mathematics decreases as a result of the 'fun mathematics'.

3.5.2. The Participating Teachers

The comparative research of how two different teachers attain the intentions of a teaching experiment (Fyhn, submitted manuscript) gives rise to the question of whether they believe in what they claim independent of the researcher or whether the researcher has seduced them to make their claims.

There is a risk that the teachers who participated in the climbing days were more or less seduced to claim that they believe the climbing approach to angles is a valuable way of teaching. If so, the result could be that the teachers more or less attain the GSRP's intentions. But the result just as well could be that the teachers just smile politely and keep their old teaching practice.

This is a question about research ethics: in cases where the researcher needs to get involved with particular motivated participants there is great risk of manipulating them. In such cases any kind of persuasion needs careful consideration.

One guide for identifying persuasion is to separate the teachers' statements into what they are acquainted with and what they just know by description (Russel, 1963). If they claim that they believe in something they know just by description, there is a risk that this is something they are persuaded to believe.

In the Gorgias Dialogue (Plato, 1968) Socrates claimed that there are two sorts of persuasion, one which is the source of belief without knowledge, and the other one is of knowledge. This way of having knowledge about an object is quite similar to what Russel (1963) denotes as to be acquainted with an object; the sort of persuasion which is the source of belief that includes knowledge; the sort of persuasion that leads to conviction.

The teachers who take part in the angles-and-climbing lessons are acquainted with climbing and with mathematics teaching. But mathematics teaching based on climbing is a kind of teaching that they knew only by description when the experiment started.

4. The Four Papers and the Two DVDs in the GSRP

4.1. A Survey

Figure 1 presents how the four papers in this study are related to each other. Paper one (Fyhn, 2007; 2004) is a pilot study which investigates the angle-as-turn approach to angles in the compass context. The analyses concluded that a new context was needed and that led to paper two (Fyhn, 2006) which is a case study with focus on angles in a climbing context.

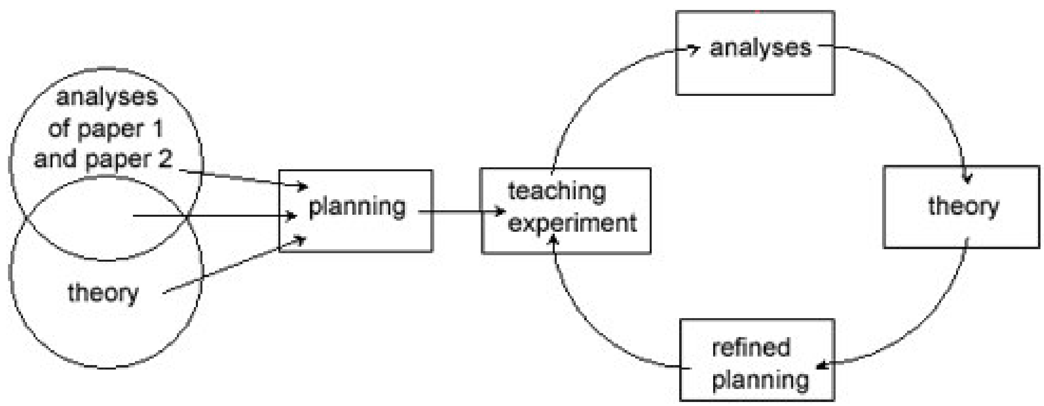

Figure 1. Relations between the four papers in the GSRP. The cycle in the right part of the figure is described in paper three. Paper four analyses whether the participating teachers have attained the GSRP intentions.

DVD 2 presents some of the findings from paper three and this DVD is primarily meant for teachers. The analyses of paper four (Fyhn, submitted manuscript) concluded that perhaps teachers need to be familiar with inductive enactive teaching before they are able to grasp the intentions of the GSRP. The DVD 1, which is based upon the analyses from paper one, can function as an introduction to inductive enactive teaching.

Both the students in paper one (Fyhn, 2007) and in paper three (Fyhn, accepted for publication a) performed a pre-test and a post-test. The angle task in figure 2 was included in both of the pre-tests and the angle task in figure 3 was included in both of the post-tests.

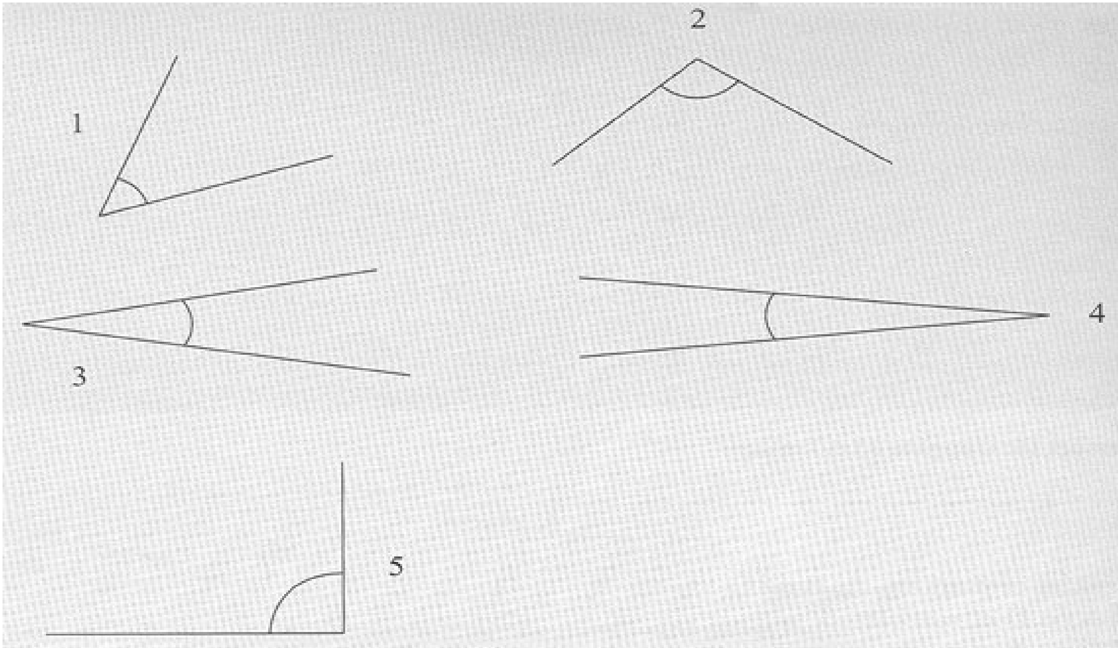

Figure 2. The angle task given in the pre-test. Questions: a) *Which one of the marked angles do you believe is the largest one?* b) *Which one of the marked angles do you believe is the smallest one?*

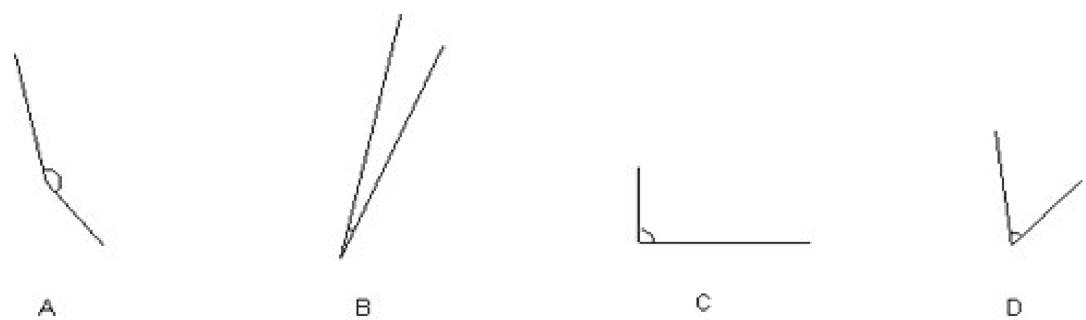

Figure 3. The angle task given in the post-test. Questions: a) *Which one of the marked angles do you believe is the largest one?* b) *Which one of the marked angles do you believe is the smallest one?*

4.2. Paper one

The curriculum's (KUF, 1996a; KUF, 1996b) focus on the turn approach to angles was the main reason for emphasizing the compass as context in the teaching of angles. Paper one (Fyhn, 2007) concluded that the compass probably is less useful as an approach to angles for students who are not sure about how angles are to be measured. This finding supports Mitchelmore and White (2000) who conjectured that angles are easiest to grasp for children when both of the sides are visible. The study in paper one (Fyhn, 2007) seems to be a good example that motivated students and an engaged teacher who intends to follow the curriculum are not necessarily enough for the students' learning.

The use of the compass in the teaching of angles is a deductive approach. The compass itself is a heavily mathematised deductive tool; it is a detailed instrument that points out the direction towards the Magnetic North Pole. Probably the compass can function as a useful tool for students who have reached the third level (2.5.7) and who are looking for new ways to express their understanding.

4.3. DVD 1

DVD 1 (Fyhn, unpublished DVD) is meant for teachers who are not used to basing their teaching upon students' own physical experiences. Findings from paper four (Fyhn, submitted manuscript) indicate that teachers need to experience inductive tasks where the mathematics is obviously visible before they are able to mathematise what Niss (1999) denotes as " complex extra-mathematical contexts". Thus DVD 1 is meant for teachers who need knowledge of how students' inductive enactive experiences can be used as a basis for mathematics teaching (4.6; Fyhn, submitted manuscript).

DVD 1 presents two practical activities in mathematics together with some written follow-up tasks. Both the activities and the tasks focus on relations between a circle's circumference and its radius. This focus is important for the students because Norwegian students are introduced to 'angle-as-measure' before they have met the unit circle.

According to Lakoff and Núñez (2000), numbers are often conceptualised as points on a line. In Norwegian primary school, students are familiar with using one-dimensional scales to measure lengths, weight, time and other subjects. Angles, however, cannot spontaneously be measured this way, because there is no immediate correlation between an angle, Φ, and the length of the arc subtended by Φ (ibid).

The theoretical framework for DVD 1 (Fyhn, unpublished DVD) is Dienes' stages. Because the students performed the activities as part of a two day trip to the mountains, the first stage was naturally integrated in the students' work. Most of the film-clips concern inductive work on stage two. The third and fourth stages focus on the transition from mesospace work to work on paper; the two dimensional figures on paper mean 'the same' as the figures made in the snow.

Most of the follow-up tasks in the DVD concern work on stages four and five. The fifth stage is divided into two, the first part focuses on oral language while the second part focuses on written language. The DVD intends to support the students' progress towards the stages four and five because they can return to the lower stages whenever they want by referring to what they actually performed themselves.

4.4. Paper two

In the second paper (Fyhn, 2006) the 'rotation' approach to angles is put aside, favouring the 'shape' approach. In addition, the angles are treated in a new context; outdoor life is replaced by climbing. This is a single case study of a twelve-year-old student who was particularly motivated towards climbing. The results from these analyses (ibid.) indicated that the 'shape approach' to angles in a climbing context could be worth trying on an entire class.

The researcher visited some friends just after they had made a successful climbing trip; the parents and two daughters who were twelve and fifteen years old had ascended a classic climbing route in the nearby region. The ascendance of a climbing route could be interpreted to be what Freudenthal (1973) denotes as geometry, as grasping the space in which you live, breath and move. Thus the main research question in paper two (Fyhn, 2006) was:

> ***Is a climbing discourse a possible resource for a school-geometry discourse?***

In this single case study a girl was guided by the researcher to discover several angles in her own narrative from a climbing trip. Before telling her story, the girl claimed that it concerned neither angles nor geometry. The girl's use of 'angle' is analysed within two frameworks; Foucault's (2004) discourse theory and Lakoff and Núñez' (2000) metaphor theory. The paper concluded that the climbing discourse could be a possible discourse for the teaching of angles.

4.5. Paper three

The analyses in paper two (Fyhn, 2006) supported by Mitchelmore and White (1998) gave direct rise to a new teaching experiment, a project where the girl's class participated in two days where *angle* and *climbing* were in focus. The first day was a climbing day while day two was a follow-up day at school. Paper three's (Fyhn, accepted for publication a) aim is to develop theory that can lead to an improved design of the teaching experiment. The paper focuses on the participating students during these two days of work. The research question is

> ***How do students describe and explain angles in drawings and written text when they mathematise climbing with respect to angles?***

At the end of day two the students were to make a drawing from the climbing walls in their own way and point at something concerning angles in the figure. These drawings were categorised as narrative or analytical. All the narrative drawings were made by girls, actually both the two girls who succeeded with the pre-test task in figure 2 made nice narrative overviews. The analytical drawings were made by both boys and girls. The narrative drawings turned out to show either horizontal mathematising or no mathematising at all while five of the six analytical drawings were interpreted to show both horizontal and vertical mathematising.

The students' writings were analyzed with respect to different levels of understanding. These levels were quite close to the Van Hiele (1986) levels. It turned out that level three was more difficult to identify than the first two levels. This could be explained by the strong Norwegian measure approach to the teaching of angles. The logical relations between acute, right and obtuse angles belong to level three; this is the level before measurement is introduced. According to Van Hiele definitions belong to the third level (ibid.); an acute angle is defined to be smaller than a right angle while an obtuse angle is defined to be larger than a right angle.

The refined plan in figure 1 has two central points where the first one is to focus on analytical drawing from the very beginning and to point out the difference between analytical and narrative drawing. The second point is to pay attention to each of the refined levels which has its own language:

Level 1: The word 'angle' is related to recognition of angles.

Level 2: Angles' shapes are described for instance by the words 'acute', 'right' and 'obtuse'.

Level 3: Angles' sizes are explained for instance by the words 'acute', 'right' and 'obtuse'. Statements about how some angle's size or change of size can decide how hard it is to ascend a climbing route.

4.6. DVD 2

The idea of DVD 2 (Fyhn, accepted for publication b) is to introduce teachers to a theory level of students' learning. On this DVD the same climbing sequence is presented four times; first as a survey and then with focus on each of the three first refined GSRP levels of understanding. The idea is to show how the same activity can be used in work on different levels; teachers here can be introduced to what Treffers (1987) denotes as vertical planning of teaching.

Research results too often do not reach teachers and this DVD is an attempt to decrease the gap between research results and teachers.

4.7. Paper four

A new way of teaching is of limited value unless some teachers start to practice it and paper four (Fyhn, submitted manuscript) is a comparative case study of two different teachers who took an active part in the teaching experiment. The paper (ibid.) describes and analyses how the two participating teachers attain the intentions of the GSRP;

> ***How do teachers attain students' mathematising of climbing as an approach to their teaching of angles?***

The paper discusses the ethical aspect of doing some exciting activity with a class and then making teachers claim whether they believe in this teaching. The teachers turned out to have different intentions than the researcher. In addition, the text intends to reveal whether and to what extent the teachers are seduced by the researcher to make their claims.

The students and the teachers participated in one more climbing-and-angles day three months after the two days described in paper three (Fyhn, accepted for publication a) and the research focus is to search for and identify eventual similarities in the two different teachers' written utterances. The analyses showed that there were some similar claims from the two very different teachers and similar results from different informants can generate theory (Andersen, 2003).

The findings indicate that the two different teachers underwent a similar development towards attaining the GSRP's intentions. Firstly, appreciating the inductive mesospace approach and secondly, discovering the mathematising approach. Because the teachers here met three new aspects in teaching, they needed time to grasp the project's intentions.

However, the teachers experienced taking part in the GSRP as participating in developmental work because most of the analyses of the students' work were not worked out before afterwards. Perhaps the teachers' responses to the results of these analyses would have influenced their claims as well as their attainments of the GSRP's intentions.

The analyses in paper four (Fyhn, submitted manuscript) indicate that the teachers needed to get acquainted with inductive enactive mesospace teaching before they were able to grasp the students' mathematising of climbing. This gives rise to future research: "Is inductive enactive teaching experience necessary for teachers to grasp students' mathematising?"

5. The Climbing Approach to Angles in a Broader Perspective

5.1. Guided Reinvention of Angles

Freudenthal (1983) asks if the historical learning process of mankind somehow can be repeated by individual learners and he continues by claiming that no individual needs to run through the historical pedigree and conceptual hierarchy of knowledge and abilities.

> Learners should be allowed to find their own levels and explore the paths leading there with as much and as little guidance as each particular case requires (ibid., p. 47).

The fact that Euclid's' definition of angles regarded angles that both were larger than the zero angle and smaller than the straight angle combined with the origin of the word *angle,* indicates that the bent shapes of the students' own bodies could be a proper starting point for their work with angles in primary school.

Hanson (1958, p. 15) claims that "Seeing is not only the having of a visual experience; it is also the way in which the visual experience is had." The introduction of angles in a context which most students experience as exciting and fun prepares for a positive attitude towards mathematics:

> If the students experience the process of reinventing mathematics as expanding common sense, then they will experience no dichotomy between everyday life experience and mathematics. Both will be part of the same reality (Gravemeijer and Doorman, 1999, p. 127).

5.2. Inductive and deductive approaches

According to Freudenthal (1973) the custom of starting geometry with principles like definitions, postulates and axioms is at least one century older than Euclid's Elements. And this strong Greek deductive tradition continued to influence geometry teaching even in the 19^{th} century (ibid.).

The Van Hiele levels (Van Hiele, 1986) demand a clear inductive approach to students' development of understanding. Deductive thinking is actually excluded from the two lowest levels. This leads to the conjecture that if or when a teacher tries to implement the levels in her or his teaching, the teacher is guided towards an inductive approach to teaching.

Frameworks like the Van Hiele levels quite often are seen as belonging to the domain of researchers (Pegg, Gutiérrez and Huerta, 1998). One attempt to reach teachers with a similar framework is DVD 2 'Angles in Climbing' (Fyhn, accepted for publication b).

In work with students at the first two periods (Van Hiele, 1986) the discussions will consist of open-ended questions that will be answered by gestures, drawings, words and writings. The lack of closed questions can prevent the teachers from falling into their more or less well-established deductive habits.

5.2.1. Physics

The earliest idea of climbing as an approach to the teaching of angles was founded on the researcher's background from rock climbing in her younger days; climbing was analysed as practical examples of mechanics and Fyhn (2006) concluded that climbing concerned vectors.

According to Doorman (2005) mathematics and physics are related disciplines that attempt to describe phenomena in physical and mathematical terms in order to deal with them in a sensible way and "classifying an activity as physical or as mathematical is difficult" (ibid., pp. 4-5).

Van Hiele (1988) asks for an investigation to analyse the levels in physics. The climbing approach to angles can be treated as an early inductive approach to the teaching of vectors. Vectors are central elements in biomechanics which is a discipline within physics. Central concepts in climbing are gravity, friction, forces, balance and pendulum, all of which concern physics.

The climbing context can function as a basis for work with vectors in physics and mathematics in upper secondary school and then the Van Hiele levels can be further developed in physics.

5.3. To See Something *as* Something else

Regarding design research, Gravemeijer and Cobb (2006) claim that a theoretical analysis is the result of a purposeful and complex problem-solving process. Different researchers are therefore not expected to develop identical theoretical constructs from analysing the same set of data. "The paradigm observer is not the man who sees and reports what all normal observers see and report, but the man who sees in familiar objects what no one else has seen before" (Hanson 1958, p. 30).

Through the analyses in paper three (Fyhn, accepted for publication a) the Van Hiele levels had to become modified to some extent because the Van Hiele theory concerned geometrical figures while the GSRP concerned angles related to one particular context.

The complex problem-solving process that lead to the analyses of paper three (ibid.) concluded that the third level focuses on logical relations between angles and climbing. Angles are implicit in the climbers' analyses of their moves and the GSRP level three focuses on making this understanding explicit, but without creating a new 'climbing-language'.

The GSRP design and analyses were strongly influenced by two points that could restrain the development of the design; a) the wish to avoid the unsuccessful but thorough implemented Norwegian measure approach to angles and b) the guided re-invention goal “to get long chains of long-term learning processes” (Freudenthal, 1991, p. 66). The GSRP was influenced by these two points in the following way, a) the focus intended to be the understanding of angles and not their sizes and b) an underlying goal was the understanding of vectors.

5.3.1. Logical Relations between Angles

Boswinkel and Moerland (2006) support Van Hiele (1986) in claiming that achieving understanding in mathematics is a process that develops through different levels and that the formal level is based upon the understanding at the lower levels. Boswinkel and Moerland (2006) use the ice-berg as metaphor in underlining how the formal level rests upon the lower levels of understanding as shown in figure 4.

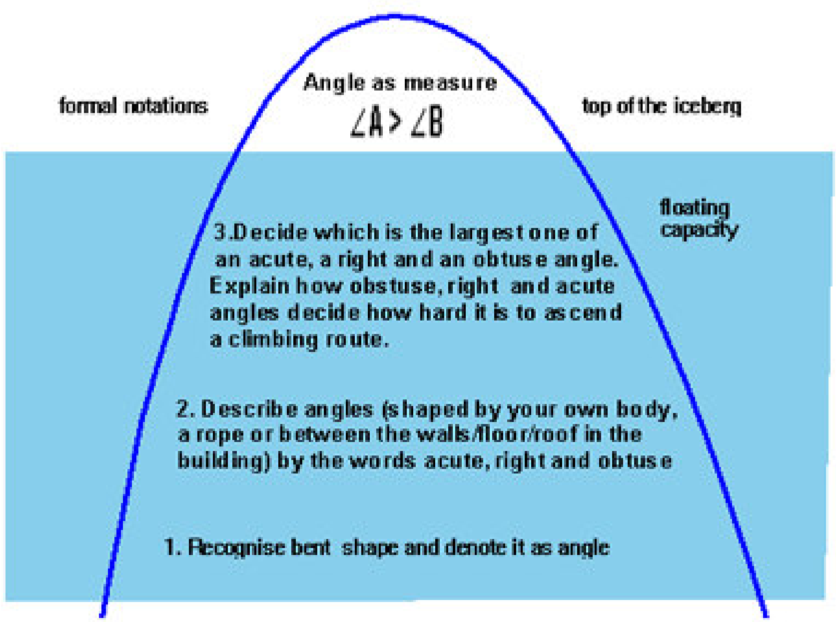

Figure 4. The Ice-berg. Based upon Boswinkel and Moerlands (2006)

The study of this metaphor served as a guide to see the third level as how acute, right and obtuse angles decide how hard it is to ascend a climbing route. However, before that arguing can take place the student needs to know which one is the largest of an acute, a right and an obtuse angle.

At the second level the words acute, right and obtuse were just used to describe angles while the words acute, right and obtuse are not even part of the vocabulary at the first level.

A logical relation between two angles can be to judge whether one angle is larger or smaller than another one, without necessarily measuring the angles' sizes in degrees. Freudenthal (1983) claims that students must be able to distinguish between equal or different angles before measuring angles take place. The analyses of students' writings and drawings in paper three (Fyhn, accepted for publication a) showed that after the teaching experiment was completed several students still struggled with judging which one was the largest one among two angles.

5.3.1.1. Beyond Freudenthal?

One of the girls made a correct statement regarding logical relations between climbing and angles: "…when you climb you become more tired if your arms are held in a 90° angle than if they are stretched out" (ibid.). This girl turned out to fail on the angle task in the post-test. One interpretation of this is that the girl knew what a 90° angle looked like; she was familiar with the visual gestalt 90° angle but she did not know whether an obtuse angle was larger or smaller than 90°. That led to the following assumption: Students must be able to judge which one of two different angles is the largest one, before measuring angles is introduced.

5.3.2. The Norwegian Curriculum

The ice-berg metaphor can probably function as a strong tool in underlining how the basic inductive work is the fundament for the deductive formal work. Boswinkel and Moerland (2006) distinguish between three informal (lower) levels: 1) Basal level, 2) Model level and 3) Building blocks level. The ice-berg in figure 4 is an 'angle' version of Boswinkel and Moerland's (ibid.) ice-berg. Actually the four ice-berg levels are interpreted to be similar to the first four Van Hiele levels.

According to the Norwegian Curriculum of 2006 (KD, 2006a), one of the teaching goals is that students in fourth grade should be able to estimate and measure angles. The GSRP analyses clearly indicate that this aim will probably not be achieved; to introduce students in fourth grade to measuring angles to some extent is similar to forgetting that the top of the ice-berg is dependent on what is underneath the water surface. Thus the GSRP supports the idea of making Norwegian curriculum design more based upon research.

5.4. Angles in Teaching and Learning

5.4.1. Angle as a Noun and Angle as a Verb

According to Barton (1999) mathematics is a way of talking; a circle is an ideal object that exists because we talk about it. “In those languages where roundness is embodied as an action, not as an object, circles no not exist” (ibid., p. 56). As presented in section 2.1.1 *angle* is both a verb and a noun in the Norwegian daily language but in Norwegian mathematics language *angle* is reduced to being just a noun.

5.4.1.1. To Angle and to Bend

The GSRP has pointed at possibilities for including angle as a verb in the work with angles throughout the first three levels, as part of the floating capacity presented in figure 4. Perhaps more ways of using *angle* as a verb can be developed in Norwegian mathematics teaching.

One problem with using the verb to ‘angle’ instead of to ‘bend’ is that the more you bend something the less is its angle. This fact can be treated as a golden possibility for challenging students’ possible misconceptions regarding angles. However, the students need to have reached the second level before this task can give meaning to them; the language of this task belongs to the third period (Van Hiele, 1986).

5.4.1.2. The Metaphor to ‘Angle’

The verb to ‘angle’ means to present (something) from a particular point of view (Henriksen and Haslerud, 2002). This indicates that to ‘angle’ has an established position as metaphor because “Language and the meanings conveyed by language do not come out of thin air” (Lakoff and Núñez, 2000, p. 347).

To ‘angle’ something from a different point of view means to present it from a different point of view. This use of ‘angle’ can lead to the misconception that an angle is a slanting line; an angle has only one side. Perhaps this metaphorical use of angle was the reason why the girl claimed that there was an angle between two points (Fyhn, 2006). A group discussion in the classroom focusing on the meaning of the metaphor to ‘angle’ is a task that can be used for challenging students’ possible misconceptions regarding angles.

5.4.2. The Refined GSRP Levels

The revised plan for the teaching of angles follows the four levels presented in figure 4.

> Level 1. Recognition of angles; angle as bent shape. Students at this level are able to recognise a bent shape and denote it as *angle*.

Level 2. Description of angles. Students at this level are able to describe angles and denote them as *acute angle, right angle* and *obtuse angle*.

Level 3. Angles' sizes and their consequences. Students at this level are able to decide whether one angle (for instance acute angle) is *smaller than*, *equal to* or *larger than* another angle (for instance right angle) and to make statements about how some particular angle's size can decide how hard it is to ascend a climbing route.

Level 4. Formal level. Students at this level area are able to measure angles and to refer to angles' sizes in degrees. They are also able to use written notations regarding angles and to *write down formal statements* that claim whether one angle is smaller than, equal to or larger than another one.

The work with angles could start with students working in pairs in the classroom; one of them bends her or his body while the other one shall recognise this bent shape. The students shall draw stick-men and mark the angles on these figures.

Next the students are introduced to climbing at a climbing arena. Students here work in groups of three or four persons. One student climbs at a time and the other ones makes stick-man drawings that intend to represent the climber. Through this work the students are asked to give oral descriptions of angles shaped by the climber's body and then they are introduced to the words acute, obtuse and right.

When a student has grasped the words acute, right and obtuse, the next step is which of them is the largest one. Questions to discuss are a) what does it mean that one angle is larger or smaller than another one? And b) what is the easiest way to climb, with a small or large angle between the upper arm and the forearm? Here the students' drawings and their experiments with their own bodies can function as useful elements in the discussion.

After the climbing the students make figures of stick-men where they formally denote angles as $\angle A$, $\angle B$, $\angle$ PQR and so on and supply their drawings by writing down formal statements like $\angle A < \angle B$. When the students have reached this formal level they probably have a fundament for being introduced to *degrees* as a unit for referring to angles' sizes.

5.4.3. Further Use of the Climbing Approach to Angles

DVD 2 has been used in January and February 2007 for teachers at two primary schools with totally 105 students at 6^{th} grade. The teachers prepared their students, guided them through a angles-in-climbing lesson at a climbing wall and finally carried out follow-up work at school (5.4.2). 98 students were present and all of them participated in the climbing.

One trouble with further use of the climbing approach to angles is money; because most schools do not have a climbing wall the students need to rent a climbing wall and climbing equipment for performing a climbing lesson. There also is a need for extra persons to ensure the students' security while they are climbing. Finally, all these extra resources lead to logistic difficulties; it is much easier to carry out teaching inside the ordinary school building with the ordinary student groups.

Time is one more factor that prevents teachers from using the climbing as approach to the teaching of angles. Because there is a need for transport to and from the climbing wall and because there is focus on the context as well as on the mathematics, a relatively large amount of time is needed for carrying out the climbing approach to angles. However, teachers who have not grasped the difficulties many students have in grasping the angle concepts can not be expected to accept that the teaching of angles really needs to take time.

5.4.4. The Role of the Teachers

Paper four ended with a question that points towards further research, "Is inductive enactive teaching experience necessary for teachers to grasp students' mathematising?" (Fyhn, submitted manuscript, p. 26). This question needs enlightening in order to find out whether experiences from performing enactive teaching are a criterion for teachers who will take part in further development of the GSRP design.

DVD 1 intends to function as a gate for teachers who want to develop their own geometry teaching; maybe it is easier for teachers to grasp the students' mathematising in DVD 2 after having worked out the DVD 1 content first.

5.5. Gender, Angles and Physical Activity

5.5.1. Narrative and Analytical Expressions

Three gender differences were found in the analyses of paper three (Fyhn, accepted for publication a). First: The students who did not succeed with the angle task in the post test were all girls. Second: The girls who succeeded with the angle task in the post test were all interpreted to write narrative texts at the end of day two, and most of them made narrative drawings as well. No boys were interpreted to deliver a narrative work here - and the girls who made analytical work failed on the angle task in the post test. Third: One boy got special lessons due to his reading and writing difficulties and among the boys he wrote most lines of text about his expectations and experiences from the climbing day. About half of the girls wrote more lines than him. However, the number of participating students is rather small.

One possible interpretation of this finding is that the 'clever' girls had been encouraged by their art and woodwork teachers to make narrative drawings. In Norwegian schools there seems to be no tradition for making drawings as part of the students' work in geometry and thus maybe the students had no experience of the difference between analytical and narrative drawings. Murphy and Elwood (1998, p. 174) point out, "Teachers seemed to reward and encourage narrative and descriptive writing over and above factual and analytical work."

These findings can be interpreted such that the traditional mathematics teaching does not focus on students' analytical written work. A nearby conjecture is that the 'clever' girls do not learn about the difference between analytic and narrative writing; they are encouraged to write narratively by their language teachers. Furthermore, because the 'clever' girls are not taught the difference between a narrative and an analytical drawing, their narrative drawings to a less extent can support their mathematical reasoning.

5.5.2. Gender and Physical Activity

During the period from twelve until sixteen years of age, more boys than girls participate regularly in physical activities in their leisure time (Wold et al., 2000). So when students' use of their own body functioned as a basis for teaching there was a need for finding a physical activity that was expected to appeal to the girls.

Hansen (2005) focuses on girls' decreasing interest in daily physical activity during lower secondary school. He indicates seven considerations that can give girls more positive experiences from daily physical activities at school: less competition, increased character of play, more training of skills, possibilities to choose a wanted activity, more gender separation, and smaller groups (ibid., author's translation).

Climbing fulfils these considerations to a large extent and actually all of the girls who took part in the climbing lessons reported in paper three (Fyhn, accepted for publication a) reported that they enjoyed the climbing. This finding supports the climbing approach to angles; the gymnastics teachers here have good reason for supporting and taking part in the climbing lessons together with the mathematics teachers.

5.6. Angles in Space

According to Lakoff and Núñez (2000), space can be conceptualised in two different ways, natural continuous space and space as a set of points. Natural continuous space is the space in which we live our three-dimensional lives; actually this space is such a natural part of our daily lives that we do not question its existence. Descartes invented Space-as-a Set- of-Points (ibid.) and by doing so he algebraized geometry (Freudenthal, 1973).

Mathematicians move back and forth between these two conceptions of space and it takes training to think in terms of Space-as-a Set- of-Points (Lakoff and Núñez, 2000).

5.6.1. Two and Three Dimensions

The concept of dimension can be difficult to grasp for two reasons, "Dimension implies direction, implies measurement…" (Appelbaum, 1992, p. viii). Dimension here is interpreted to belong to Space-as-a-Set-of-Points (Lakoff and Núñez, 2000). But this does not automatically imply that it is similarly difficult to grasp a concept when it is introduced in natural continuous space.

A student's own body in natural continuous mesospace appears as more concrete than a two dimensional figure on paper that intends to be a representation of this person's body, even if this figure is a photo. Such a two-dimensional photo cannot become more concrete than the three-dimensional person her- or himself; "Images, reflexions, pictures and maps in fact copy originals with different degrees of strictness" (Hanson, 1958).

If the teaching of a concept (angle) in geometry starts in three-dimensional mesospace, then the teacher needs to consider the complete path the students need to pass in order to move from their concrete real experiences (climbing) and to their pictorial descriptions and linguistic explanations of the geometrical concept (angle) that are related to these experiences (climbing). Both teachers and students will probably benefit from Hanson's (1958) words:

> We must explore the gulf between pictures and language, between sketching and describing, drawing and reporting….
>
> …picturing and speaking are different…and brought together they must be if observations are to be significant and noteworthy (ibid., p. 25).

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